Graduate Texts in Mathematics

John B.Conway

A Course in Functional Analysis

Second Edition

泛函分析教程 第2版

Springer-Verlag
世界图书出版公司

John B. Conway

A Course in Functional Analysis

Second Edition

Springer

书　　名：A Course in Functional Analysis 2nd ed.
作　　者：J. B. Conway
中 译 名：泛函分析教程 第 2 版
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：010-64015659, 64038347
电子信箱：kjsk@vip.sina.com
开　　本：24　　　印　张：17.5
出版年代：2003 年 6 月
书　　号：7-5062-5951-6/ O・370
版权登记：图字：01-2003-3771
定　　价：39.00 元

John B. Conway
Department of Mathematics
University of Tennessee
Knoxville, Tennessee 37996
USA

With 1 Illustration.

Mathematical Subject Classification (2000): 46-01, 46L05, 47B15, 47B25

Library of Congress Cataloging-in-Publication Data
Conway, John B.
A course in functional analysis/John B. Conway.—2nd ed.
p. cm. — (Graduate texts in mathematics; 96)
Includes bibliographical references.
ISBN 0-387-97245-5 (alk. paper)
1. Functional analysis. I. Title. II. Series.
QA320.C658 1990
515.7—dc20 90-9585

Printed on acid-free paper.

Reprinted in China by Beijing World Publishing Corporation, 2003

9 8 7 6

ISBN 0-387-97245-5
ISBN 3-540-97245-5 SPIN 10869896

Springer-Verlag New York Berlin Heidelberg
A member of BertelsmannSpringer Science+Business Media GmbH

For Ann (of course)

Preface

Functional analysis has become a sufficiently large area of mathematics that it is possible to find two research mathematicians, both of whom call themselves functional analysts, who have great difficulty understanding the work of the other. The common thread is the existence of a linear space with a topology or two (or more). Here the paths diverge in the choice of how that topology is defined and in whether to study the geometry of the linear space, or the linear operators on the space, or both.

In this book I have tried to follow the common thread rather than any special topic. I have included some topics that a few years ago might have been thought of as specialized but which impress me as interesting and basic. Near the end of this work I gave into my natural temptation and included some operator theory that, though basic for operator theory, might be considered specialized by some functional analysts.

The word "course" in the title of this book has two meanings. The first is obvious. This book was meant as a text for a graduate course in functional analysis. The second meaning is that the book attempts to take an excursion through many of the territories that comprise functional analysis. For this purpose, a choice of several tours is offered the reader—whether he is a tourist or a student looking for a place of residence. The sections marked with an asterisk are not (strictly speaking) necessary for the rest of the book, but will offer the reader an opportunity to get more deeply involved in the subject at hand, or to see some applications to other parts of mathematics, or, perhaps, just to see some local color. Unlike many tours, it is possible to retrace your steps and cover a starred section after the chapter has been left.

There are some parts of functional analysis that are not on the tour. Most authors have to make choices due to time and space limitations, to say nothing of the financial resources of our graduate students. Two areas that are only briefly touched here, but which constitute entire areas by themselves, are topological vector spaces and ordered linear spaces. Both are beautiful theories and both have books which do them justice.

The prerequisites for this book are a thoroughly good course in measure and integration—together with some knowledge of point set topology. The appendices contain some of this material, including a discussion of nets in Appendix A. In addition, the reader should at least be taking a course in analytic function theory at the same time that he is reading this book. From the beginning, analytic functions are used to furnish some examples, but it is only in the last half of this text that analytic functions are used in the proofs of the results.

It has been traditional that a mathematics book begin with the most general set of axioms and develop the theory, with additional axioms added as the exposition progresses. To a large extent I have abandoned tradition. Thus the first two chapters are on Hilbert space, the third is on Banach spaces, and the fourth is on locally convex spaces. To be sure, this causes some repetition (though not as much as I first thought it would) and the phrase "the proof is just like the proof of..." appears several times. But I firmly believe that this order of things develops a better intuition in the student. Historically, mathematics has gone from the particular to the general—not the reverse. There are many reasons for this, but certainly one reason is that the human mind resists abstraction unless it first sees the need to abstract.

I have tried to include as many examples as possible, even if this means introducing without explanation some other branches of mathematics (like analytic functions, Fourier series, or topological groups). There are, at the end of every section, several exercises of varying degrees of difficulty with different purposes in mind. Some exercises just remind the reader that he is to supply a proof of a result in the text; others are routine, and seek to fix some of the ideas in the reader's mind; yet others develop more examples; and some extend the theory. Examples emphasize my idea about the nature of mathematics and exercises stress my belief that doing mathematics is the way to learn mathematics.

Chapter I discusses the geometry of Hilbert spaces and Chapter II begins the theory of operators on a Hilbert space. In Sections 5–8 of Chapter II, the complete spectral theory of normal compact operators, together with a discussion of multiplicity, is worked out. This material is presented again in Chapter IX, when the Spectral Theorem for bounded normal operators is proved. The reason for this repetition is twofold. First, I wanted to design the book to be usable as a text for a one-semester course. Second, if the reader understands the Spectral Theorem for compact operators, there will be less difficulty in understanding the general case and, perhaps, this will lead to a greater appreciation of the complete theorem.

Chapter III is on Banach spaces. It has become standard to do some of this material in courses on Real Variables. In particular, the three basic principles, the Hahn-Banach Theorem, the Open Mapping Theorem, and the Principle of

Uniform Boundedness, are proved. For this reason I contemplated not proving these results here, but in the end decided that they should be proved. I did bring myself to relegate to the appendices the proofs of the representation of the dual of L^p (Appendix B) and the dual of $C_0(X)$ (Appendix C).

Chapter IV hits the bare essentials of the theory of locally convex spaces—enough to rationally discuss weak topologies. It is shown in Section 5 that the distributions are the dual of a locally convex space.

Chapter V treats the weak and weak-star topologies. This is one of my favorite topics because of the numerous uses these ideas have.

Chapter VI looks at bounded linear operators on a Banach space. Chapter VII introduces the reader to Banach algebras and spectral theory and applies this to the study of operators on a Banach space. It is in Chapter VII that the reader needs to know the elements of analytic function theory, including Liouville's Theorem and Runge's Theorem. (The latter is proved using the Hahn–Banach Theorem in Section III.8.)

When in Chapter VIII the notion of a C^*-algebra is explored, the emphasis of the book becomes the theory of operators on a Hilbert space.

Chapter IX presents the Spectral Theorem and its ramifications. This is done in the framework of a C^*-algebra. Classically, the Spectral Theorem has been thought of as a theorem about a single normal operator. This it is, but it is more. This theorem really tells us about the functional calculus for a normal operator and, hence, about the weakly closed C^*-algebra generated by the normal operator. In Section IX.8 this approach culminates in the complete description of the functional calculus for a normal operator. In Section IX.10 the multiplicity theory (a complete set of unitary invariants) for normal operators is worked out. This topic is too often ignored in books on operator theory. The ultimate goal of any branch of mathematics is to classify and characterize, and multiplicity theory achieves this goal for normal operators.

In Chapter X unbounded operators on Hilbert space are examined. The distinction between symmetric and self-adjoint operators is carefully delineated and the Spectral Theorem for unbounded normal operators is obtained as a consequence of the bounded case. Stone's Theorem on one parameter unitary groups is proved and the role of the Fourier transform in relating differentiation and multiplication is exhibited.

Chapter XI, which does not depend on Chapter X, proves the basic properties of the Fredholm index. Though it is possible to do this in the context of unbounded operators between two Banach spaces, this material is presented for bounded operators on a Hilbert space.

There are a few notational oddities. The empty set is denoted by $\square$. A reference number such as (8.10) means item number 10 in Section 8 of the present chapter. The reference (IX.8.10) is to (8.10) in Chapter IX. The reference (A.1.1) is to the first item in the first section of Appendix A.

There are many people who deserve my gratitude in connection with writting this book. In three separate years I gave a course based on an evolving set of notes that eventually became transfigured into this book. The students

in those courses were a big help. My colleague Grahame Bennett gave me several pointers in Banach spaces. My ex-student Marc Raphael read final versions of the manuscript, pointing out mistakes and making suggestions for improvement. Two current students, Alp Eden and Paul McGuire, read the galley proofs and were extremely helpful. Elena Fraboschi typed the final manuscript.

John B. Conway

Preface to the Second Edition

The most significant difference between this edition and the first is that the last chapter, Fredholm Theory, has been completely rewritten and simplified. The major contribution to this simplification was made by Hari Bercovici who showed me the most simple and elegant development of the Fredholm index I have seen.

Other changes in this book include many additional exercises and numerous comments and bibliographical notes. Several of my friends have been helpful here. The greatest contributor, however, has been Robert B. Burckel; in addition to pointing out mistakes, he has made a number of comments that have been pertinent, scholarly, and very enlightening. Several others have made such comments and this is a good opportunity to publicly thank them: G.D. Bruechert, Stephen Dilworth, Gerald A. Edgar, Lawrence C. Ford, Fred Goodman, A.A. Jafarian, Victor Kaftall, Justin Peters, John Spraker, Joseph Stampfli, J.J. Schäffer, Waclaw Szymanski, James E. Thomson, Steve Tesser, Bruce Watson, Clifford Weil, and Pei Yuan Wu.

Bloomington, Indiana
December 7, 1989

John B. Conway

Contents

CHAPTER VII
Banach Algebras and Spectral Theory for Operators on a Banach Space

CHAPTER VIII
C^*-Algebras

CHAPTER IX
Normal Operators on Hilbert Space

CHAPTER X
Unbounded Operators

CHAPTER I

Hilbert Spaces

A Hilbert space is the abstraction of the finite-dimensional Euclidean spaces of geometry. Its properties are very regular and contain few surprises, though the presence of an infinity of dimensions guarantees a certain amount of surprise. Historically, it was the properties of Hilbert spaces that guided mathematicians when they began to generalize. Some of the properties and results seen in this chapter and the next will be encountered in more general settings later in this book, or we shall see results that come close to these but fail to achieve the full power possible in the setting of Hilbert space.

§1. Elementary Properties and Examples

Throughout this book $\mathbb{F}$ will denote either the real field, $\mathbb{R}$, or the complex field, $\mathbb{C}$.

1.1. Definition. If $\mathscr{X}$ is a vector space over $\mathbb{F}$, a *semi-inner product* on $\mathscr{X}$ is a function u: $\mathscr{X} \times \mathscr{X} \to \mathbb{F}$ such that for all α, β in $\mathbb{F}$, and x, y, z in $\mathscr{X}$, the following are satisfied:

(a) $u(\alpha x + \beta y, z) = \alpha u(x, z) + \beta u(y, z)$,
(b) $u(x, \alpha y + \beta z) = \bar{\alpha} u(x, y) + \bar{\beta} u(x, z)$,
(c) $u(x, x) \geqslant 0$,
(d) $u(x, y) = \overline{u(y, x)}$.

Here, for α in $\mathbb{F}$, $\bar{\alpha} = \alpha$ if $\mathbb{F} = \mathbb{R}$ and $\bar{\alpha}$ is the complex conjugate of α if $\mathbb{F} = \mathbb{C}$. If $\alpha \in \mathbb{C}$, the statement that $\alpha \geqslant 0$ means that $\alpha \in \mathbb{R}$ and α is non-negative.

Note that if $\alpha = 0$, then property (a) implies that $u(0, y) = u(\alpha \cdot 0, y) =$

$\alpha u(0, y) = 0$ for all y in $\mathscr{X}$. This and similar reasoning shows that for a semi-inner product u,

(e) $u(x, 0) = u(0, y) = 0$ for all x, y in $\mathscr{X}$.

In particular, $u(0, 0) = 0$.

An *inner product* on $\mathscr{X}$ is a semi-inner product that also satisfies the following:

(f) If $u(x, x) = 0$, then $x = 0$.

An inner product in this book will be denoted by

$$\langle x, y \rangle = u(x, y).$$

There is no universally accepted notation for an inner product and the reader will often see (x, y) and $(x | y)$ used in the literature.

1.2. Example. Let $\mathscr{X}$ be the collection of all sequences $\{\alpha_n : n \geqslant 1\}$ of scalars α_n from $\mathbb{F}$ such that $\alpha_n = 0$ for all but a finite number of values of n. If addition and scalar multiplication are defined on $\mathscr{X}$ by

$$\{\alpha_n\} + \{\beta_n\} \equiv \{\alpha_n + \beta_n\},$$
$$\alpha\{\alpha_n\} = \{\alpha\alpha_n\},$$

then $\mathscr{X}$ is a vector space over $\mathbb{F}$.

If $u(\{\alpha_n\}, \{\beta_n\}) \equiv \sum_{n=1}^{\infty} \alpha_{2n}\bar{\beta}_{2n}$, then u is a semi-inner product that is not an inner product. On the other hand,

$$\langle \{\alpha_n\}, \{\beta_n\} \rangle = \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n,$$

$$\langle \{\alpha_n\}, \{\beta_n\} \rangle = \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n \bar{\beta}_n,$$

$$\langle \{\alpha_n\}, \{\beta_n\} \rangle = \sum_{n=1}^{\infty} n^5 \alpha_n \bar{\beta}_n,$$

all define inner products on $\mathscr{X}$.

1.3. Example. Let (X, Ω, μ) be a measure space consisting of a set X, a σ-algebra Ω of subsets of X, and a countably additive measure μ defined on Ω with values in the non-negative extended real numbers. If f and $g \in L^2(\mu) \equiv L^2(X, \Omega, \mu)$, then Hölder's inequality implies $f\bar{g} \in L^1(\mu)$. If

$$\langle f, g \rangle = \int f\bar{g}\, d\mu,$$

then this defines an inner product on $L^2(\mu)$.

Note that Hölder's inequality also states that $|\int f\bar{g}\,d\mu| \leqslant [\int |f|^2 d\mu]^{1/2} \cdot [\int |g|^2 d\mu]^{1/2}$. This is, in fact, a consequence of the following result on semi-inner products.

1.4. The Cauchy–Bunyakowsky–Schwarz Inequality. *If $\langle\cdot,\cdot\rangle$ is a semi-inner product on $\mathscr{X}$, then*

$$|\langle x,y\rangle|^2 \leqslant \langle x,x\rangle\langle y,y\rangle$$

for all x and y in $\mathscr{X}$. Moreover, equality occurs if and only if there are scalars α and β, both not 0, such that $\langle \beta x+\alpha y, \beta x+\alpha y\rangle = 0$.

Proof. If $\alpha \in \mathbb{F}$ and x and $y \in \mathscr{X}$, then

$$\begin{aligned} 0 &\leqslant \langle x-\alpha y, x-\alpha y\rangle \\ &= \langle x,x\rangle - \alpha\langle y,x\rangle - \bar{\alpha}\langle x,y\rangle + |\alpha|^2\langle y,y\rangle. \end{aligned}$$

Suppose $\langle y,x\rangle = be^{i\theta}$, $b \geqslant 0$, and let $\alpha = e^{-i\theta}t$, t in $\mathbb{R}$. The above inequality becomes

$$\begin{aligned} 0 &\leqslant \langle x,x\rangle - e^{-i\theta}tbe^{i\theta} - e^{i\theta}tbe^{-i\theta} + t^2\langle y,y\rangle \\ &= \langle x,x\rangle - 2bt + t^2\langle y,y\rangle \\ &= c - 2bt + at^2 \equiv q(t), \end{aligned}$$

where $c = \langle x,x\rangle$ and $a = \langle y,y\rangle$. Thus $q(t)$ is a quadratic polynomial in the real variable t and $q(t) \geqslant 0$ for all t. This implies that the equation $q(t) = 0$ has at most one real solution t. From the quadratic formula we find that the discriminant is not positive; that is, $0 \geqslant 4b^2 - 4ac$. Hence

$$0 \geqslant b^2 - ac = |\langle x,y\rangle|^2 - \langle x,x\rangle\langle y,y\rangle,$$

proving the inequality.

The proof of the necessary and sufficient condition for equality is left to the reader. ■

The inequality in (1.4) will be referred to as the CBS inequality.

1.5. Corollary. *If $\langle\cdot,\cdot\rangle$ is a semi-inner product on $\mathscr{X}$ and $\|x\| \equiv \langle x,x\rangle^{1/2}$ for all x in $\mathscr{X}$, then*

(a) $\|x+y\| \leqslant \|x\| + \|y\|$ *for x, y in $\mathscr{X}$,*
(b) $\|\alpha x\| = |\alpha|\,\|x\|$ *for α in $\mathbb{F}$ and x in $\mathscr{X}$.*

If $\langle\cdot,\cdot\rangle$ is an inner product, then

(c) $\|x\| = 0$ *implies* $x = 0$.

Proof. The proofs of (b) and (c) are left as an exercise. To see (a), note that for x and y in $\mathscr{X}$,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + \langle y,x\rangle + \langle x,y\rangle + \|y\|^2 \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2. \end{aligned}$$