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Second Edition

# Classical and Computational Solid Mechanics

Y. C. Fung • Pin Tong  
Xiaohong Chen

 World Scientific

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# **Classical and Computational Solid Mechanics**

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# **Classical and Computational Solid Mechanics**

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*by Y. C. Fung, Pin Tong and Xiaohong Chen*

## DEDICATION

To Luna, Conrad, and Brenda

Y. C. Fung

To my teacher Professor Fung and to Mrs. Fung who have been the constant source of caring, inspiration and guidance; to my parents who were the source of love and discipline; to my wife who has been the source of love, companionship and encouragement; to my children and grandchildren who have been the source of joy and love; to Denny who will be forever in my heart.

Pin Tong

To all my wonderful teachers and mentors who pass the torch of wisdom from generation to generation.

Xiaohong Chen



## PREFACE TO THE SECOND EDITION

The objective of this book is to offer students of science and engineering a concise, general, and easy-to-understand account of some of the most important concepts and methods of classical and computational solid mechanics. The classical part is mainly a re-issue of Fung's *Foundations of Solid Mechanics*, with a major addition to the modern theories of plasticity, and a major revision of the theory of large elastic deformation with finite strains. The computational part consists of five new chapters, which focus on numerical methods to solve many major linear and nonlinear boundary-value problems of solid mechanics.

We hold the principle of easy-to-understand for the readers as an objective of our presentation. We believe that to be easily understood, the presentation must be precise, the definitions and hypotheses must be clear, the arguments must be concise and with sufficient details, and the conclusion has to be drawn very carefully. We strive to pay strict attention to these requirements. We believe that the method must be general and the notations should be unified. Hence, we presented the tensor analysis in general coordinates, but kept the indicial notations for tensors in the first twelve chapters. In Chapters 13–22, however, the dyadic notations of tensors and the notations for matrix operations are used to shorten the formulas.

This book was written for engineers who invent and design things for human kind and want to use solid mechanics to help implement their designs and applications. It was written also for engineering scientists who enjoy solid mechanics as a discipline and would like to help develop and advance the subject further. It was further designed to serve those physical and natural scientists and biologists and bioengineers whose activities might be helped by classical and computational solid mechanics. For example, biologists are discovering that the functional behavior of cells depends on the stresses acting on the cell. It is widely recognized that the molecular mechanics of the cell must be developed as soon as possible.

Solid mechanics deals with deformation and motion of “solids.” The displacement that connects the instantaneous position of a particle to its position in an “original” state is of general interest. The preoccupation about particle displacements distinguishes solid mechanics from that of fluids.

This book begins with an introductory chapter containing a brief sketch of history, an outline of some prototypes of theories, and a description of some



more complex features of solid mechanics. In Chapter 2, an introduction to tensor analysis is given. The bulk of the text from Chapters 3 to 12 is concerned with the classical theory of elasticity, but the discussion also includes the thermodynamics of solid, thermoelasticity and plasticity. Chapters 13–16 extend the discussion to finite deformation theory, viscoelasticity, viscoplasticity, coupled thermal, mechanical and electric processes in thermodynamic nonequilibrium based on functional and/or state-variable approaches and to their applications to electro-thermo-viscoelastic/plastic problems. Fluid mechanics basically is excluded, but methods that are common to both fluid and solid mechanics are emphasized. Both dynamics and statics are treated; the concepts of wave propagation are introduced in an early stage. Variational calculus is emphasized since it provides a unified point of view and is useful in formulating approximate theories and computational methods. The large deflection theory of plates presented in the concluding section of Chapter 13 illustrates the elegance of the general approach to the large deformation theory.

Chapters 17 to 22 are devoted to computational solid mechanics to deal with linear, nonlinear, and inhomogeneous problems. Chapter 17 develops the incremental theory in considerable detail. It is recognized that the incremental approach is the most practical approach. Chapter 18 is devoted to numerical methods, with the finite element singled out for detailed discussion and its application to elasticity. Chapter 19 presents the calculation methods based on the mixed and hybrid variational principles, illustrating the broadening of the computation power with less restrictive (or weaker) hypotheses in formulating the variational principle. Chapter 20 deals with finite element methods of plates and shells, making the computation methods accessible to the analysis of the structures of aircraft, marine architectures, land vehicles, and shell-like structures in human beings, animals, plants, earth, and space. Chapter 21 deals with finite element modeling of nonlinear elasticity, viscoelasticity, plasticity, viscoplasticity and creep. Finally the book concludes with Chapter 22 on the Meshless Local Petrov–Galerkin and Eshelby–Atluri Methods, alternatives to the conventional finite element methods. Thus, a broad sweep of modern, advanced topics are covered. Since Chapters 17–21 are independent of Chapters 13–16, readers may skip the latter in the first reading.

Overall, this book lays emphasis on general methodology. It prepares the students to tackle new problems. However, as it was said in the original preface of the *Foundations of Solid Mechanics*, no single path can embrace the broad field of mechanics. As in mountain climbing, some routes are safe to travel, others more perilous; some may lead to the summit, others to different vistas of interest; some have popular claims, others are less traveled. In choosing a

particular path for a tour through the field, one is influenced by the curriculum, the trends in literature, and the interest in engineering and science. Here, a particular way has been chosen to view some of the most beautiful vistas in classical and computational mechanics. In making this choice, we have aimed at straightforwardness and interest, and practical usefulness in the long run.

Holding the book to a reasonable length did not permit inclusion of many numerical examples, which have to be supplemented through problems and references. Fortunately, there are many excellent references to meet this demand. We have presented an extensive bibliography in this book, but we suggest that the reader consult the review journal *Applied Mechanics Reviews* (AMR) published by the American Society of Mechanical Engineers International since 1947, for current information. The reader is referred to the periodic in-depth reviews of the literature in specific issues of AMR.

We are indebted to many authors and colleagues as acknowledged in the preface of the *Foundations of Solid Mechanics*. In the preparation of the present edition, we are especially indebted to Professors Satya Atluri and Theodore Pian. We would like to record our gratitude to many colleagues who wrote us to discuss various points and sent us errata in the *Foundations of Solid Mechanics* and *Classical and Computational Solid Mechanics*, especially to Drs. Satya N. Atluri, Pao-Show D. Cheng, Shun Cheng, Ellis H. Dill, Clive L. Dym, J. B. Haddow, Manohar P. Kamat, Hans Krumhaar, T. D. Leko, Howard A. Magrath, Sumio Murakami, Theodore Pian, R. S. Rivlin, William P. Rodden, Bertil Storakers, Howard J. White, Jr. and John C. Yao. We would also like to thank Professors Y. Ohashi, S. Murakami and N. Kamiya for translating the *Foundations* book into Japanese, Professors Oyuang Zhang, Ma Wen-Hua and Wang Kai-Fu for translating the *Foundations* book into Chinese.

We would like to take this opportunity to mention a few of editorial notes:

- (1) The bibliography is given at the end of the book.
- (2) Equations in each section are numbered sequentially. When referring to equations in other sections, we use the format (Sec.no:Eq.no), e.g. (5.6:4).
- (3) Formulas are a concise way of saying lots of things. The most important formulas are marked with a triangular star,  $\blacktriangle$ . They are worthy of being committed to memory.

Yuan-Cheng Fung  
Pin Tong  
Xiaohong Chen



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