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Harmonic Analysis

调和分析

Elias M. stein

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Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals

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with the assistance of Timothy S. Murphy

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Preface

Since the publication of the previous books in this series more than twenty years ago,[†] the subject of harmonic analysis has undergone a vast development. A succession of new departures has in many ways transformed the whole field. It has brought with it additional insights, extending significantly the range of previous ideas. With all of this has come a further clarification of the essential unity linking several of the main areas of analysis.

What has been achieved includes: the broadening of the scope of real-variable methods to encompass, among other matters, the theory of Hardy spaces; the further study of the Fourier transform leading to summability and restriction theorems; the analysis of L^2 methods, exploiting notions of orthogonality and oscillatory integrals; the application of these ideas to situations where geometric properties related to curvature play a role; and other ramifications of some of these concepts in the context of the Heisenberg group. These ideas and their consequences have not only had a profound effect on the domain of Fourier analysis *per se* but have also had a major influence in such areas as partial differential equations, several complex variables, and analysis on symmetric spaces.

It is the objective of this book to try to give an account of the main lines of these developments. Given the sweep of the subject, I think of the task involved as being in some ways akin to the telling of a long and complicated story: an epic tale stretching over several decades, involving various principal characters that appear and reappear (sometimes in disguised form), and following a complex plot with a number of intricate subplots. To pursue this analogy further, I cannot deny that this book is in part autobiographical: as the narrator of the story, I have chosen to recount those matters I know best by virtue of having first-hand knowledge of their unravelling.

A few words about the organization of this book. It is divided into three parts, in accordance with the subject matter. The first five chapters take up real-variable theory; the next six chapters emphasize L^2 methods and oscillatory integrals; the last two chapters introduce analysis on the Heisenberg group and also provide a retrospective view of some of the

[†] See Stein and Weiss [1971] and Stein [1970c], cited in the bibliography.

preceding material. The exposition is guided by the desire that, in each chapter, the material should be developed in the service of the proofs of a few central theorems. In doing this, I have often not formulated the final results in maximal generality, nor have I always chosen the shortest proofs. It is hoped, however, that the reader will ultimately see the advantage of this approach.

It is my pleasure to acknowledge my deep indebtedness to Timothy Murphy for indispensable help in writing this book. He joined me in this effort about three years ago; up to that time only rudimentary progress had been made. From then on, our constant conversations and his many incisive suggestions spurred the work and helped refine the material into its present form. He also took complete charge of the copy editing and typesetting of the manuscript, which he carried out with consummate skill.

I also wish to express my thanks to those others who have aided me: Daryl Geller, Fulvio Ricci, Cora Sadosky, and Christopher Sogge, who made valuable suggestions that have been incorporated in the text; also D. H. Phong, Robert Fefferman, David Jerison, Andrew Bennett, Peter Heller, Andrew Neff, and Der-Chen Chang, who prepared lecture notes of graduate courses I gave at Princeton University during the period 1972 to 1987, on which parts of this book are based.

Lastly, I wish to express my gratitude to all my students and collaborators whose ideas enlighten these pages. This book is in large measure a record of their achievement.

Elias M. Stein
December 1992

Guide to the Reader

The core of the book, which appears in standard type, consists of the thirteen chapters, excluding the appendices and sections titled "Further Results". Written to be as self-contained as possible, its object is to present the main ideas without undue adornment. In addition, the following features should be noted.

Appendices are intended as elaborations of previously treated core subjects and are given with substantial sketches of proofs.

Further Results are meant to survey the vast number of additional extensions and cognate topics; these are often presented as mere reformulations of theorems in the cited literature, sometimes also with some indication of proof.

Previous monographs in this series† are cited from time to time for helpful background material. A number of interesting related topics, not pursued in this book, can also be found there, as well as earlier, more elemental (and possibly more transparent) approaches to some of the matters treated here.

† Stein and Weiss [1971] and Stein [1970c]; in the sequel, these books will be referred to simply as *Fourier Analysis* and *Singular Integrals*.

Contents

Harmonic Analysis

Real-Variable Methods, Orthogonality, and Oscillatory Integrals

Prologue

Given the complexity of the matters treated here, it may be helpful to begin by giving an overview of our subject. In sketching its broad outlines, we point first to the principal analytic constructs whose study will be our chief concern. These concepts can be loosely grouped into three categories: maximal averages, singular integrals, and oscillatory integrals.

Like all deep ideas in mathematics, these have each taken several forms, displaying their versatility by adapting to the changing contexts in which they occurred. Let us briefly recall how each appeared in an early version.

Maximal averages. The simplest instance arises when we consider the family of averages of a function f on $\mathbf{R}^1$ given by $\frac{1}{2t}\int_{-t}^{t} f(x-y)\,dy$, $t > 0$, as well as the more sophisticated variant $\frac{t}{\pi}\int_{-\infty}^{\infty} \frac{f(x-y)}{y^2+t^2}\,dy$, $t > 0$, which is the Poisson integral of f. For these, the limiting behavior as $t \to 0$ is the main interest, and its deeper study is subsumed in the properties of the corresponding maximal functions.

Singular integrals. A basic object in the classical theory is the Hilbert transform $f \mapsto \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} f(x-y)\frac{dy}{y}$. Its indispensable role there is partly explained by the fact that it stands squarely at the crossroads linking real variables and complex function theory.

Oscillatory integrals. Here the primordial example is the Fourier transform $f \mapsto \int_{-\infty}^{\infty} e^{-2\pi i x\cdot\xi} f(x)\,dx$. Of course, when thinking of it, we should also have in mind its n-dimensional form, as well as the oscillatory integrals arising from this by symmetry considerations, such as Bessel functions.

Now it was already understood early that these three concepts were, to a substantial degree, intertwined. Thus the fundamental L^2 estimate for the Hilbert transform was seen as a simple consequence of the use of the Fourier transform, and the weak-type (1,1) estimate was originally proved by using properties of the Poisson integral mentioned above.

What could not be guessed then, and could only be revealed with the passage of time, were the wider and deeper interconnections inherent in these examples and their successive generalizations and refinements. The insights that this yielded provide the foundations of a theory of vast scope and utility that has developed over the last thirty years, spurred by its application to such parts of analysis as partial differential equations, several complex variables, and harmonic analysis related to semisimple Lie groups and symmetric spaces.

While the theory encompassing these ideas does not admit a brief summary, we do wish to touch on some of its main themes.

(i) *The underlying real-variable structure.* A central role in the analysis of maximal functions and singular integrals is played by the covering lemmas of Vitali and Whitney types. While this was first understood in the context of $\mathbf{R}^n$ (with its usual translation and dilation structure), significant parts of these results can be extended to much more general settings, where the analogues of these lemmas continue to hold. Moreover, as it turned out, more refined versions of the older results could be proved by examining further the techniques based on these covering arguments.

(ii) *Hardy space theory.* We comment first on the ubiquitous nature of the L^p spaces, $1 < p < \infty$. First, the pervasiveness of L^2 estimates is a basic fact of analysis, given the essential part played by the Fourier transform and other devices involving orthogonality. Second, while it might have been simpler to limit considerations to L^1 and L^∞ estimates, long experience has shown that deep and interesting assertions of this kind rarely hold. Thus the function of L^p is twofold: as a compromise of the possible; but more importantly, that the analysis it requires often reveals fundamental properties of the operators in question.

Now it is exactly with the failure of L^1 and L^∞ that Hardy space theory may be thought to begin. Originally developed in the context of one complex variable with a different emphasis in mind, in its modern incarnation this topic represents a happy culmination of the study of maximal functions and singular integrals by real variable methods. Not only does it yield a rich H^1 theory, making up for many of the shortcomings of L^1, but it also gives us a fruitful H^p theory in the case $p < 1$, where L^p was entirely barren. That H^p would seem destined to be of further interest in the future can be guessed from the fact that the most common "singularities" in analysis, such as those given by rational functions, or carried on analytic subvarieties, or representable by Fourier integral ("Lagrangian") distributions, are all locally in H^p, for some $p \leq 1$.

(iii) *More extended singular integrals.* The singular integrals alluded to so far have all been of the form

$$(Tf)(x) = \int K(x, y)\, f(y)\, dy,$$

where the singularity of the kernel $K(x, y)$ is concentrated in y near x. A significant departure of the current theory is that it can begin to come to grips with the situation that arises when the singularity is now "spread out", say for y in some variety Σ_x. When an analysis in this context is possible, orthogonality again plays a key role, sometimes via the Fourier transform, but more often using other oscillatory integrals. An important observation is that, at bottom, what makes this possible is some sort of "curvature" property of the family $\{\Sigma_x\}$. In this setting, analogues of maximal functions arise by taking averages over (proper) submanifolds of $\mathbf{R}^n$. Again, curvature properties play a decisive role in their study.

(iv) *Oscillatory integrals.* As indicated above, oscillatory integrals provide a necessary tool in exploiting the geometric properties related to curvature and orthogonality in the more extended maximal operators and singular integrals that have arisen. However, these oscillatory integrals, and others of interest, are not easily classified and come in a multiplicity of forms: variants of the Fourier transform, convolution operators (such as Bochner-Riesz means), and Fourier integral operators are among these forms. What is clear is that this part of the theory is in its infancy, and much more remains to be understood.

(v) *Heisenberg group.* The study of the Heisenberg group illustrates a number of essential ideas treated in this book. In particular, it gives an excellent example of the real-variable structure mentioned above; connected with this is the Cauchy-Szegö projection operator, which is a naturally occurring instance of a singular integral in this general context. In addition, we might point out that inherent in its structure is the notion of "twisted convolution"; it accounts for the composition formula for pseudo-differential operators (in their symmetric form) and also yields important examples of oscillatory singular integrals. But beyond these didactic uses the significance of the Heisenberg group resides in what it has allowed us to do, namely, to explore the way into the broader applications of our subject to such interesting areas as several complex variables and (subelliptic) partial differential equations.

CHAPTER I

Real-Variable Theory

We begin by setting down some of the fundamental real-variable ideas behind the theory of the maximal operator and the boundedness of singular integrals. To proceed here requires that our underlying space be endowed with a certain kind of metric structure. The model for this is $\mathbf{R}^n$, equipped with its usual family of Euclidean balls, which is the setting appropriate for the standard translation-invariant theory.† In fact, by abstracting some simple and basic features of this case (connected with the covering lemmas of Vitali and Whitney), a number of key points of the earlier development can be carried out in a much broader context. The following additional comments may be helpful in placing the subject of this chapter in its proper perspective.

(i) We prove here the weak-type (1,1) and L^p inequalities for the maximal operator in the generality alluded to above. We also deal with the corresponding facts for singular integrals. However, for the latter our results are of a conditional nature, since they depend on an additional assertion (essentially the L^2 boundedness) that must be treated separately. In the translation-invariant case, this is exactly where the Fourier transform is decisive. In our general context, other notions must also come into play, but consideration of these aspects is postponed until they are systematically taken up in chapters 6 and 7.

(ii) What will be even clearer (in later chapters) is that maximal operators and singular integrals can be thought of as part of a threefold unity, in that these two operators are intimately tied to another construct, namely that of square functions. One way to realize this unity is to consider all three as singular integrals, but now as vector-valued versions taking their values in differing Banach spaces.

(iii) When we continue beyond this chapter we shall not feel constrained by the requirement to present matters in the generality used here. Instead, for simplicity of exposition, we shall usually content ourselves with the standard setting of $\mathbf{R}^n$, and invoke the general theory only when needed in particular circumstances. Instances where the general point of view plays an important role are the weighted inequali-

† As developed in, e.g., *Singular Integrals*, chapters 1 and 2.