AN INTRODUCTION TO

SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Classical and Variational Solutions

Doina Cioranescu
Patrizia Donato
Marian P. Roque

World Scientific

The book extensively introduces classical and variational partial differential equations (PDEs) to graduate and post-graduate students in Mathematics. The topics, even the most delicate, are presented in a detailed way. The book consists of two parts which focus on second order linear PDEs. Part I gives an overview of classical PDEs, that is, equations which admit strong solutions, verifying the equations pointwise. Classical solutions of the Laplace, heat, and wave equations are provided. Part II deals with variational PDEs, where weak (variational) solutions are considered. They are defined by variational formulations of the equations, based on Sobolev spaces. A comprehensive and detailed presentation of these spaces is given. Examples of variational elliptic, parabolic, and hyperbolic problems with different boundary conditions are discussed.

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To all those who will read this book. May it give them some insight in the fascinating field of partial differential equations.
To Ion and our son Jean-Michel with love.

Doina Cioranescu

To the peacebuilder Daisaku Ikeda and mentor of my life, with deepest gratitude.
To my husband Eric and sons Maikel and Johendry, persons of great value, with all my love.

Patrizia Donato

To my parents, Martin and Consolacion, brothers Marlon and Marius, husband Rolly, and daughter Tin-Tin, with love and gratitude for having supported all my endeavors. Unlike partial differential equations, their love requires no conditions.

Marian P. Roque

Preface

The study of partial differential equations (PDEs) is at the crossroads of mathematical analysis, measure theory, topology, differential geometry, scientific computing, and many other branches of mathematics. Modeling physical phenomena, partial differential equations are fascinating topics because of their increasing presence in treating real physical processes. In recent years, PDEs have become essential tools when modeling in fields such as materials science, fluid mechanics, quantum mechanics, mathematical finance, biology and biomedicine, and environmental sciences.

The aim of this book is to introduce this important subject to graduate and post-graduate students in Mathematics with sufficient background in advanced calculus, real analysis, and functional analysis, as well as to young researchers in the field. Indeed, as explained below, various reading levels are possible, and because of that we also hope it will be useful to colleagues teaching PDEs at different levels.

The book is essentially devoted to second order linear partial differential equations and consists of two parts that are each self-contained. Part I (Chapters 1 to 5) gives a comprehensive overview of classical PDEs, that is, equations which admit smooth (strong) solutions. Part II (Chapters 6 to 10) deals with variational PDEs, where weak solutions are considered. They are defined via a weak (variational) formulation of the equations and are searched in suitable function spaces (here in particular, Sobolev spaces). These spaces, being the essential tools in the treatment of variational PDEs, are introduced and extensively detailed.

The first chapter answers the basic question: What is a PDE? In this chapter, we give the basic definitions encountered in the study of PDEs, the general notation used throughout the book and a list of some classical partial differential equations. Chapter 2 presents the classification of PDEs and their canonical forms. Characteristic curves and some existence theorems are also given. The classical examples of Laplace, heat, and wave equations are introduced in Chapters 3, 4, and 5, respectively. Part I is aimed to be an introductory presentation of the subject, it is why we choose not to include too many details but to state only the main methods and results, with proofs for some theorems.

For the study of weak solutions, a review of L^p-spaces and distributions is necessary. This is done in Chapter 6 where we also prove some less classical results, which are specifically needed in the succeeding chapters. A comprehensive and detailed discussion of Sobolev spaces and Sobolev continuous and compact embeddings is presented in Chapters 7 and 8, respectively. Examples of variational elliptic problems with different boundary conditions are discussed in Chapter 9. Finally, variational parabolic and hyperbolic problems are studied in Chapter 10.

The Sobolev spaces theory provides the foundation for variational PDEs. For young mathematicians, understanding the proofs can be comforting and undoubtedly formative. We decided to prove almost all the important results we stated, for completeness and for a deeper understanding of the theory underlying variational PDEs. While writing this book, we observed that some of the proofs we wanted to present in these chapters are not usually found in a PDE book, since they involve advanced topics in functional analysis. We hope that their presence in this book will be appreciated by the reader. We also give several references for further reading in every chapter.

For Chapters 7 to 10, we have chosen to explain proofs in detail to enable young researchers to do an independent study on the topics covered. We also completed the presentation by an important number of examples, in particular for the most delicate definitions.

According to the authors' experience, the material covered here is more than sufficient for a one-semester graduate course and may be extended to two or three semesters, depending on the level of the students. For an introductory course, we suggest a detailed discussion of Chapters 1 and 2 followed by solutions of the Laplace, heat, and wave equations in $\mathbb{R}^2$ from Chapters 3, 4, and 5. Definitions and theorems (without proofs) from Chapters 6, 7 and 8, together with the Lax-Milgram Theorem, can then be given before presenting some examples of variational elliptic problems from

Chapter 9. Sections and proofs skipped in the first course, can be discussed in detail in a second course. A more advanced course could contain the proofs of the most delicate results from Chapter 7 as well as those of the Sobolev embedding theorems. Then one could present the results of the eigenvalue problems from Chapter 9, and apply them to the study of the variational evolution problems done in Chapter 10.

Finally, let us mention that this book is based on the long experience of the authors as researchers and teachers in the field of PDEs, teaching both in their home universities and in research schools abroad. And as significant, the book is founded on the scientific collaborations and deep friendship between the authors, which have been enriched through the years. The first two authors have a scientific collaboration for about thirty years, and the collaboration with the third author started more than twelve years ago, when they gave a graduate PDE course for three years at the Institute of Mathematics, UP Diliman under the European Asia Link IMAMIS program. In particular, the lecture notes of this course were published in 2012, under the name "Introduction to classical and variational partial differential equations", by the University of the Philippines Press.

The present book is a more developed and detailed version of these lecture notes.

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Marian P. Roque

List of Symbols

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