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Robert Friedman

Algebraic Surfaces and Holomorphic Vector Bundles

代数曲面和全纯向量丛

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Preface

This book is based on courses given at Columbia University on vector bundles (1988) and on the theory of algebraic surfaces (1992), as well as lectures in the Park City/IAS Mathematics Institute on 4-manifolds and Donaldson invariants. The goal of these lectures was to acquaint researchers in 4-manifold topology with the classification of algebraic surfaces and with methods for describing moduli spaces of holomorphic bundles on algebraic surfaces with a view toward computing Donaldson invariants. Since that time, the focus of 4-manifold topology has shifted dramatically, at first because topological methods have largely superseded algebro-geometric methods in computing Donaldson invariants, and more importantly because of the new invariants defined by Seiberg and Witten, which have greatly simplified the theory and led to proofs of the basic conjectures concerning the 4-manifold topology of algebraic surfaces. However, the study of algebraic surfaces and the moduli spaces of bundles on them remains a fundamental problem in algebraic geometry, and I hope that this book will make this subject more accessible. Moreover, the recent applications of Seiberg-Witten theory to symplectic 4-manifolds suggest that there is room for yet another treatment of the classification of algebraic surfaces. In particular, despite the number of excellent books concerning algebraic surfaces, I hope that the half of this book devoted to them will serve as an introduction to the subject. There are few references to the general subject of vector bundles on algebraic varieties beyond the book by Okonek, Schneider and Spindler on vector bundles on projective spaces, the Astérisque volume of Seshadri on bundles over curves, and a recent book by Huybrechts and Lehn. I hope that combining the study of surfaces with that of vector bundles on them (and on curves) will be mutually beneficial to both fields. For example, detailed knowledge of a surface X is necessary in order to give a detailed picture of the moduli space of bundles over X, and results about ruled surfaces are an ingredient in the proof of the Bogomolov inequality presented here. On the other hand, the Bogomolov inequality gives important information about linear systems on surfaces, by a theorem of Reider, and in particular gives a short proof of Bombieri's theorem on the behavior of $|nK_X|$ when X is a minimal surface of general type. The original motivation of computing Donaldson invariants has however disappeared except for a brief discussion in Chapter 8 for elliptic surfaces.

It is a pleasure to thank the audience at the lectures which served as the raw material for this book, as well as David Gomprecht, my course assistant for the Park City institute, for an excellent job in proofreading the rough draft of the first part of this book. I would also like to thank Tomás Gómez and Titus Teodorescu for comments on various manuscript versions, and Dave Bayer for doing an excellent job with the figures.

New York, New York Robert Friedman

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Introduction

The study of algebraic surfaces is by now over one hundred years old. Many of the fundamental results were established by the Italian school of algebraic geometry, for example Castelnuovo's criterion for a surface to be rational (1895), the theorem of Enriques that a surface is rational or ruled if and only if P_4 or P_6 is zero (1905), and in general the role of the canonical divisor in the classification of surfaces. This theory was reworked from the modern perspective of sheaves, cohomology, and characteristic classes in a series of papers by Kodaira (1960–1968) and by the Shafarevich seminar (1961–1963). In particular, new ideas were developed to attack those questions in the classification theory which had proved resistant to the synthetic techniques of the Italian school, for example the classification of elliptic surfaces or the structure of the moduli space of $K3$ surfaces and its relationship with the period map. Another deep result which seems to be inaccessible to the classical methods is the Bogomolov-Miyaoka-Yau inequality $c_1^2 \leq 3c_2$. Moreover, the new methods could be extended to the study of compact complex surfaces (Kodaira) or algebraic surfaces in positive characteristic (Mumford and Bombieri-Mumford). Despite the great progress in understanding algebraic surfaces, many open questions remain. For example, the fundamental problem of whether there exists a classification scheme of some sort for surfaces of general type seems to require a completely new insight.

By contrast, the study of holomorphic vector bundles on algebraic surfaces is much more recent, and effectively dates back to two papers by Schwarzenberger (1961). For the case of algebraic curves, Grothendieck (1956) showed that every holomorphic vector bundle over $\mathbf{P}^1$ is a direct sum of line bundles (a result known in a different language to Hilbert, Plemelj and Birkhoff, and prior to them to Dedekind and Weber). Atiyah (1957) classified all vector bundles over an elliptic curve and made some preliminary remarks concerning vector bundles over curves of higher genus. In 1960, the picture changed radically when Mumford introduced the notion of a stable or semistable vector bundle on an algebraic curve and used geometric invariant theory to construct moduli spaces for all semistable vector bundles over a given curve. Soon thereafter Narasimhan and Seshadri (1965) related the notion of stability to the existence of a unitary flat structure (in the case of trivial determinant) or equivalently a flat connection compatible with an appropriate Hermitian metric. For curves, much recent work has centered on the enumerative geometry of the moduli space of curves. Explicit geometric constructions for the moduli space were given for genus 2 curves by Narasimhan and Ramanan (1969) and for hyperelliptic curves in general by Desale and Ramanan (1976).

In this context, Schwarzenberger made the following contributions to the theory of vector bundles over a surface. In general, for a variety X of dimension greater than 1, a vector bundle on X is not a direct sum of line bundles or an extension of line bundles. Schwarzenberger's first paper studied rank 2 bundles V which are not simple ("almost decomposable" in his terminology), in other words for which the automorphism group is larger than $\mathbb{C}^*$. He showed, using the existence of a rank 1 endomorphism on V, that V is an extension by a line bundle of a coherent sheaf of the form $L \otimes I_Z$, where L is a line bundle and Z is a 2-dimensional local complete intersection subscheme, and in the case of surfaces X he gave a mechanism for describing the set of all such extensions with a fixed Z. To do so, he passed to a blowup $\tilde{X}$ of X in order to be able to replace I_Z by a line bundle of the form $\mathcal{O}_{\tilde{X}}(-\sum_i a_i E_i)$, where the E_i are the components of the exceptional divisor and the a_i are nonnegative integers. As part of the study, he analyzed when a vector bundle on $\tilde{X}$ is the pullback of a bundle on X.

In Schwarzenberger's second paper, he showed that every rank 2 vector bundle on a smooth surface X is of the form $\pi_* L$, where $\pi\colon Y \to X$ is a smooth double cover of X and L is a line bundle on Y. He then applied this construction to construct bundles on $\mathbb{P}^2$ which were not almost decomposable; these turn out to be exactly the stable bundles on $\mathbb{P}^2$. He showed further that, if V is a stable rank 2 vector bundle on $\mathbb{P}^2$, then the Chern classes for V satisfy the basic inequality $c_1(V)^2 < 4c_2(V)$.

In the years after Schwarzenberger's papers, the study of bundles over surfaces diverged into two streams. In the first, there were various attempts to generalize Mumford's definition of stability to surfaces and higher-dimensional varieties and to use this definition to construct moduli spaces of vector bundles. Takemoto (1972, 1973) gave the straightforward generalization to higher-dimensional (polarized) smooth projective varieties that we have simply called stability here (this definition is also called Mumford-Takemoto stability, μ-stability, or slope stability). Aside from proving boundedness results for surfaces, he was unable to prove the existence of a moduli space with this definition (and in fact it is still an open question whether the set of all semistable bundles forms a moduli space in a natural way). Shortly thereafter, Gieseker (1977) introduced the notion of stability now called Gieseker stability or Gieseker-Maruyama stability. Gieseker showed that the set of all Gieseker semistable torsion free sheaves on a fixed algebraic surface X (modulo a suitable equivalence relation) formed a projective variety, containing the set of all Mumford stable vector bundles as a Zariski open set. This result was generalized by Maruyama (1978) to the case where X has arbitrary dimension. The differential geometric meaning of Mumford stability is the Kobayashi-Hitchin conjecture, that every stable vector bundle has a Hermitian-Einstein connection, unique in an appropriate sense. This result, the higher-dimensional analogue of the theorem of Narasimhan and Seshadri, was proved by Donaldson (1985) for surfaces, by Uhlenbeck and Yau (1986) for general Kähler manifolds, and also by Donaldson (1987) in the case of a smooth projective variety. (The easier converse, that an irreducible Hermitian-Einstein connection defines a holomorphic structure for which the bundle is stable, was established previously by Kobayashi and Lübke.) The geometric meaning of Gieseker stability is more mysterious, although Leung (1993) has obtained results in this direction. A related general result is Bogomolov's inequality for stable vector bundles, which follows from the Donaldson-Uhlenbeck-Yau theorem as well as from various purely algebraic arguments (Bogomolov, 1977).

The other stream in studying vector bundles consists in analyzing moduli spaces for specific classes of surfaces (and perhaps specific choices of the Chern classes). The case of $\mathbb{P}^2$ and more generally $\mathbb{P}^n$ has received a great deal of attention, and moduli spaces of vector bundles on $\mathbb{P}^2$ have been described quite explicitly by the method of monads (Barth, Hulek and Maruyama). Because this subject has been well described elsewhere (see for example [117]), we do not discuss monads in this book. The case of ruled surfaces has been analyzed by Hoppe and Spindler and also by Brosius. Takemoto briefly treated the case of abelian surfaces, but the study of vector bundles (not necessarily of rank 2) over $K3$ and abelian surfaces really got off the ground with a series of papers by Mukai. This was the state of the art until about 1985, when Donaldson theory gave a powerful impetus to the study of rank 2 vector bundles over surfaces. We shall describe some of the developments arising after 1985 at the end of Chapter 10.

There is perhaps a third stream which should be mentioned, that of the enumerative geometry of the moduli space. By now these questions have been well studied for bundles over curves (Verlinde formula, cohomology ring of the moduli space), and in some sense Donaldson theory is simply a question about the enumerative geometry of the moduli space of bundles over a surface. Deep structure theorems and conjectures in gauge theory, due to Kronheimer and Mrowka and Witten, suggest that there is a very simple enumerative structure to this moduli space, but as yet there is no way to see why this should be true purely within the context of algebraic geometry.

The goal of this book is to provide a unified introduction to the study of algebraic surfaces and of holomorphic vector bundles on them. I have tried to keep the prerequisites to a good working knowledge of Hartshorne's book on algebraic geometry [61] as well as standard commutative algebra (see for example Matsumura's book [87]). Aside from what is contained in [61], we freely use the exponential sheaf sequence on a complex manifold and the Leray spectral sequence (typically when it degenerates) as well as basic properties of Chern classes which are summarized in Chapter 2, and for which Fulton's book [45] is a standard reference. For the most part, we use the Riemann-Roch theorem only for vector bundles on a curve or surface, for which proofs are given in the exercises to Chapter 2. However, we use the Grothendieck-Riemann-Roch theorem once in Chapter 8 and the Riemann-Roch theorem for a divisor on a threefold in Chapter 10, without recalling the general statements. There is also a brief appeal to relative duality in Chapter 7 and to the existence of a relative Picard scheme for smooth fibrations of relative dimension 1 in Chapter 9. The appendix to Chapter 9 uses a little Galois theory, and some results which are not used in the rest of the book use standard facts about group cohomology. The last section of Chapter 4 assumes some basic familiarity with differential geometry on a complex manifold, for example as described in the book by Griffiths and Harris [55], and can be skipped. In Chapter 8, there is a brief discussion of Donaldson invariants which motivates some of the enumerative calculations in the rest of the chapter, but which can otherwise be omitted. Of necessity, I have largely limited myself to the part of the study of vector bundles which does not involve the heavy machinery of deformation theory or geometric invariant theory; a few descriptive sections outline the main results.

For the first eight chapters, the plan has been to alternate between the study of surfaces and the study of bundles on them. This has the pedagogical advantage that, for example, vector bundles over curves are studied in Chapter 4, then used to describe ruled surfaces in Chapter 5. In Chapter 6, we use the knowledge of ruled surfaces to describe vector bundles over them, and in Chapter 9 they reappear as part of the proof of Bogomolov's inequality. Similarly, ruled surfaces are described in Chapter 5 and elliptic surfaces in Chapter 7, and the structure of the moduli space of vector bundles over such surfaces is then described in Chapters 6 and 8. I have tried to emphasize how the internal geometry of the surface is reflected in the birational geometry of the moduli space. In the last two chapters, we drop the strict division of material: Chapter 9 gives a proof of Bogomolov's inequality, which belongs to the theory of vector bundles, as well as applications to the study of linear systems (in particular pluricanonical systems) on an algebraic surface. In Chapter 10, we prove the main theorems on the classification of algebraic surfaces and outline the current state of knowledge concerning moduli spaces of rank 2 vector bundles over algebraic surfaces. The proofs of the classification results for surfaces are old-fashioned, in the sense that they do not appeal to Mori theory. On the other hand, the old-fashioned proofs may be better adapted to handling the classification of symplectic 4-manifolds. The point of view of Mori theory and the classification results for threefolds are briefly described toward the end of the chapter. Because of the way we alternate between surfaces and vector bundles, it may be a little disorienting to try to read the book chronologically, and certainly the chapters on surfaces can be, for the most part, read independently of the chapters on vector bundles. On the other hand, the later chapters on vector bundles over ruled or elliptic surfaces draw heavily on the description of the corresponding surfaces in the chapters that precede them.

Constraints of length and time dictated that many topics had to be left out. For surfaces, I would have liked to devote more time to rational and minimally elliptic singularities and to the classification of surfaces of small degree. For vector bundles, without the main tool of deformation theory, we are only able to scratch the surface of this rapidly evolving field. Because this theory does not seem to be close to a definitive state, it seems worthwhile to focus on many concrete examples.

Finally, there are many exercises at the end of each chapter, and they are an integral part of the book. In particular, many results are left to the exercises and they are frequently used in later chapters. I hope that the emphasis on examples, both in the text and the exercises, will help to serve as an introduction to this rich and beautiful field of mathematics.

1

Curves on a Surface

Introduction

In this book, unless otherwise specified, by *surface* we shall always mean a connected compact complex manifold of complex dimension 2 which is a holomorphic submanifold of $\mathbb{P}^N$ for some N. Thus, "surface" is short for smooth (connected) complex algebraic surface. By Chow's theorem, a surface is also described as the zero set in $\mathbb{P}^N$ of a finite number of homogeneous polynomials in $N+1$ variables. The study of surfaces is concerned both with the intrinsic geometry of the surface and with the geometry of the possible embeddings of the surface in $\mathbb{P}^N$. Just as with curves, we could organize this study in order of increasing complexity. In terms of the extrinsic (synthetic) geometry of a surface in $\mathbb{P}^N$, we could for instance try to study and eventually classify surfaces in $\mathbb{P}^N$ of relatively small degree. Or we could attempt to order surfaces by complexity via some intrinsic invariants, by analogy with the genus of a curve. This is the aim of the Kodaira classification, which orders surfaces by their Kodaira dimension. For this scheme, we have a fairly complete understanding of surfaces except in the case of Kodaira dimension 2, general type surfaces. We will cover the broad outlines of the general theory of surfaces. In this chapter, we will discuss the basic invariants, intersection theory and Riemann-Roch, and the structure of the set of ample divisors. In Chapter 3, we will discuss birational geometry. Chapters 5 and 7 will concern some of the main examples of surfaces: rational and ruled surfaces, $K3$ surfaces, as well as an introduction to elliptic surfaces. Finally, in Chapter 10, we shall give a general overview of the classification of algebraic surfaces.

We begin with the description of the basic numerical and topological invariants of a surface.

Invariants of a surface

A surface X is in particular a complex manifold, and always carries a canonical orientation from its complex structure. Viewing X as an oriented 4-manifold, its main topological invariants are its fundamental group $\pi_1(X,*)$, the Betti numbers $b_i(X) = b_{4-i}(X)$, and the intersection pairing on $H_2(X;\mathbb{Z})$. Here by Poincaré duality $H_2(X;\mathbb{Z}) \cong H^2(X;\mathbb{Z})$ and intersection pairing corresponds under this isomorphism to cup product from $H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z})$ to $H^4(X;\mathbb{Z}) \cong \mathbb{Z}$ by taking the canonical orientation. Over $\mathbb{R}$, the intersection pairing is specified by $b_2(X)$ and by $b_2^+(X)$, the number of positive entries along the diagonal when the form is diagonalized over $\mathbb{R}$. We also let $b_2^-(X) = b_2(X) - b_2^+(X)$. If $X = \mathbb{P}^2$ or if X is one of an unknown but finite number of surfaces of general type whose universal cover is the unit ball in $\mathbb{C}^2$, then $H^2(X;\mathbb{R}) \cong \mathbb{R}$. If X does not belong to this finite list of examples, then $H_2(X;\mathbb{R})$ is always indefinite (cf. for example [40, p. 29, Lemma 2.4]). It then follows from the classification of quadratic forms over $\mathbb{Z}$ [138], [92] that the intersection pairing on $H_2(X;\mathbb{Z})$ mod torsion is specified by its rank, signature, and type, i.e., whether or not there exists an element $\alpha \in H_2(X;\mathbb{Z})$ with $\alpha^2 \equiv 1 \mod 2$ or not. (If there exists such an α the form is *odd* or of Type I; otherwise it is *even* or of Type II.) To decide if a surface is of Type I or Type II, we use the Wu formula, which says that $\alpha^2 \equiv \alpha \cdot [K_X] \mod 2$. Here $[K_X]$ denotes the homology class associated to the canonical line bundle K_X via $c_1(K_X)$ and Poincaré duality. Thus, again by Poincaré duality, there exists an α with $\alpha^2 \equiv 1 \mod 2$ if and only if the image of $[K_X]$ in $\bar{H}_2(X;\mathbb{Z}) = H_2(X;\mathbb{Z})$ modulo torsion is not divisible by two.

There are also the holomorphic invariants of X. The most basic ones are the *irregularity* $q(X)$ of X and the *geometric genus* $p_g(X)$ of X, defined by

$$q(X) = \dim_{\mathbb{C}} H^0(X;\Omega^1_X) = \dim_{\mathbb{C}} H^1(X;\mathcal{O}_X),$$
$$p_g(X) = \dim_{\mathbb{C}} H^0(X;\Omega^2_X) = \dim_{\mathbb{C}} H^2(X;\mathcal{O}_X).$$

Thus, $q(X)$ is the number of independent holomorphic 1-forms on X and $p_g(X)$ is the number of holomorphic 2-forms on X. We note that the fact that the two different expressions above for $q(X)$ are equal follows from Hodge theory, since X is an algebraic surface over $\mathbb{C}$, and do not hold for an arbitrary compact complex surface or for a surface defined over a field of positive characteristic; in either case the "correct" definition of $q(X)$ is $\dim H^1(X;\mathcal{O}_X)$. (That the two expressions for $p_g(X)$ are equal follows from Serre duality which holds in general.) Additional invariants are given by $h^{1,1}(X) = \dim H^1(X;\Omega^1_X)$ and $c_1(X)^2 = [K_X]^2$. The relation of these invariants to the topological ones is as follows:

$$b_1(X) = 2q(X),$$

$$b_2(X) = 2p_g(X) + h^{1,1}(X),$$
$$b_2^+(X) = 2p_g(X) + 1.$$

Here the first two equalities follow by Hodge theory and the last is one form of the Hodge index theorem for a surface. We also have the Euler characteristic

$$\begin{aligned}\chi(X) &= 1 - b_1(X) + b_2(X) - b_3(X) + 1 \\ &= 2 - 2b_1(X) + b_2(X) = 2 - 4q + 2p_g(X) + h^{1,1}(X)\end{aligned}$$

and the holomorphic Euler characteristic

$$\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = 1 - q(X) + p_g(X).$$

There is also Noether's formula (in some sense a special case of the Riemann-Roch theorem for surfaces) which says that

$$c_1^2(X) + c_2(X) = 12\chi(\mathcal{O}_X),$$

or in other words that $[K_X]^2 + \chi(X) = 12(1 - q(X) + p_g(X))$. An easy manipulation of the formulas (Exercise 1) shows that Noether's formula is equivalent to the Hirzebruch signature theorem

$$b_2^+(X) - b_2^-(X) = \tfrac{1}{3}(c_1^2(X) - 2c_2(X)).$$

Beyond this there are the "higher" holomorphic invariants of X, the *plurigenera* $P_n(X) = \dim H^0(X; K_X^{\otimes n})$, defined for $n \geq 1$. Thus, $P_1(X) = p_g(X)$. It is by now well known [39] that the plurigenera are not in general homotopy or homeomorphism invariants of X. It has recently been shown via new invariants introduced by Seiberg and Witten that the plurigenera are diffeomorphism invariants of X (see for example [16] and [41]). We shall discuss some of these developments further in Chapter 10.

Divisors on a surface

We recall that a (reduced irreducible) *curve* C on X is an irreducible holomorphic subvariety of complex dimension 1. Thus, locally C is described as $\{f(z_1, z_2) = 0\}$, where f is a holomorphic function of z_1, z_2. Of course, C need not be a (holomorphic) submanifold of X; if it is we say that C is a *smooth* curve. A *divisor* D on X is a finite formal sum $\sum_i n_i C_i$ of distinct irreducible curves C_i, where the $n_i \in \mathbb{Z}$. The set of all divisors $\operatorname{Div} X$ is thus the free abelian group generated by the irreducible curves on X. The divisor D is *effective* if the $n_i \geq 0$ for all i. An effective divisor $D \neq 0$ will also be called a curve. We write $D \geq 0$ if D is effective and $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$. If f_i is a local equation for the curve C_i, then D is locally described by the meromorphic function $\prod_i f_i^{n_i}$, which is in fact holomorphic if and only if D is effective. Conversely, a meromorphic function f on X has an associated divisor (f), which is the curve of zeros of f minus the