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Jaume Llibre
Rafael Ramírez

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Jaume Llibre
Departament de Matemàtiques
Universitat Autònoma de Barcelona
Barcelona, Spain

Rafael Ramírez
Departament d'Enginyeria Informàtica
Universitat Rovira i Virgili
Tarragona, Catalonia, Spain

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We dedicate this book
to the memory of
Professor A.S. Galiullin

Preface

In the theory of ordinary differential equations we can distinguish two fundamental problems. The first, which we may call the *direct problem*, is, in a broad sense, to find all solutions of a given ordinary differential equation. The second, which we may call the *inverse problem* and which is the focus of this work, is to find the most general differential system that satisfies a given set of properties. For instance, we might wish to identify all differential systems in $\mathbb{R}^N$ that have a given set of invariant hypersurfaces or that admit a given set of first integrals.

Probably the first inverse problem to be explicitly formulated was the problem in celestial mechanics, stated and solved by Newton in Philosophiae Naturalis Principia Mathematica (1687), of determining the potential force field that yields planetary motions that conform to the motions that are actually observed, namely, to Kepler's laws.

In 1877 Bertrand [10] proved that the expression for Newton's force of attraction can be obtained directly from Kepler's first law. He also stated the more general problem of determining the positional force under which a particle describes a conic section for any initial conditions. Bertrand's ideas were developed in particular in the works [42, 51, 78, 149].

In the modern scientific literature the importance of this kind of inverse problem in celestial mechanics was already recognized by Szebehely, see [152].

In view of Newton's second law, that acceleration is proportional to the applied force, it is clear that the inverse problems just mentioned are equivalent to determining second-order differential equations based on prespecified properties of the right-hand side of the equations.

The first statement of an inverse problem as the problem of finding the most general first-order differential system satisfying a given set of properties was stated by Erugin [52] in dimension two and developed by Galiullin in [60, 61].

The new approach to inverse problems that we propose uses as an essential tool the Nambu bracket. We deduce new properties of this bracket which play a major role in the proof of all the results of this work and in their applications. We observe that the applications of the Nambu bracket that we will give in this book are original and represent a new direction in the development of the theory of the Nambu bracket.

In the first chapter we present results of two different kinds. First, under very general assumptions we characterize the ordinary differential equations in $\mathbb{R}^N$ that have a given set of M partial integrals, or a given set of $M < N$ first integrals, or a given set of $M \leq N$ partial and first integrals. Second, we provide necessary and sufficient conditions for a system of differential equations in $\mathbb{R}^N$ to be integrable, in the sense that the system admits $N-1$ independent first integrals.

Because of the unknown functions that appear, the solutions of the inverse problems in ordinary differential equations that we give in the first chapter have a high degree of arbitrariness. To reduce this arbitrariness we must impose additional conditions. In the second chapter we are mainly interested in planar polynomial differential systems that have a given set of polynomial partial integrals. We discuss the problem of finding the planar polynomial differential equations whose phase portraits contain invariant algebraic curves that are either generic (in an appropriate sense), or contain invariant algebraic curves that are non-singular in $\mathbb{RP}^2$ or are nonsingular in $\mathbb{R}^2$, or that contain singular invariant algebraic curves. We study the particular case of quadratic polynomial differential systems with one singular algebraic curve of arbitrary degree.

In the third chapter we state Hilbert's 16th problem restricted to algebraic limit cycles. Consider Σ'_n, the set of all real polynomial vector fields $\mathcal{X} = (P, Q)$ of degree n having real irreducible invariant algebraic curves (where irreducibility is with respect to $\mathbb{R}[x, y]$). A simpler version of the second part of Hilbert's 16th problem restricted to algebraic limit cycles can be stated as follows: *Is there an upper bound on the number of algebraic limit cycles of any polynomial vector field of* Σ'_n? By applying the results given in the second chapter we solve this simpler version of Hilbert's 16th problem for two cases: (a) when the given invariant algebraic curves are generic (in a suitable sense), and (b) when the given invariant algebraic curves are non-singular in $\mathbb{CP}^2$. We state the following conjecture: *The maximum number of algebraic limit cycles for polynomial planar vector fields of degree n is* $1 + ((n-1)(n-2)/2)$. We prove this conjecture for the case where n is even and the algebraic curves are generic M-curves, and for the case that all the curves are non-singular in $\mathbb{R}^2$ and the sum of their degrees is less than $n+1$.

We observe that Hilbert formulated his 16th problem by dividing it into two parts. The first part asks for the mutual disposition of the maximal number of ovals of an algebraic curve; the second asks for the maximal number and relative positions of the limit cycles of all planar polynomial vector fields $\mathcal{X} = (P, Q)$ of a given degree. Traditionally the first part of Hilbert's 16th problem has been studied by specialists in real algebraic geometry, while the second has been investigated by mathematicians working in ordinary differential equations. Hilbert also pointed out that connections are possible between these two parts. In the third chapter we exhibit such a connection through the Hilbert problem restricted to algebraic limit cycles.

In the fourth chapter, applying results of the first chapter *we state and solve the inverse problem for constrained Lagrangian mechanics*: for a given natural mechanical system with N degrees of freedom, determine the most general force field that depends only on the position of the system and that satisfies a given set of constraints linear in the velocity. One of the main objectives in this inverse problem is to study the behavior of constrained Lagrangian systems with constraints linear in the velocity in a way that is different from the classical approach deduced from the d'Alembert–Lagrange principle. As a consequence of the solution of the inverse problem for the constrained Lagrangian systems studied here we obtain the general solution for the inverse problem in dynamics for mechanical systems with N degrees of freedom. We also provide the answer to the generalized Dainelli inverse problem, which before was solved only for $N = 2$ by Dainelli. We give a simpler solution to Suslov's inverse problem than the one obtained by Suslov. Finally, we provide the answer to the generalized Dainelli–Joukovsky problem solved by Joukovsky in the particular case of mechanical systems with two or three degrees of freedom.

Chapter 5 is devoted to *the inverse problem for constrained Hamiltonian systems*. That is, for a given submanifold $\mathcal{M}$ of a symplectic manifold $\mathbb{M}$ we determine the differential systems whose local flow leaves the submanifold $\mathcal{M}$ invariant. We study two cases: (a) $\mathcal{M}$ is determined by l first integrals with $l \in [\dim \mathbb{M}/2, \dim \mathbb{M})$, and (b) $\mathcal{M}$ is determined by $l < \dim \mathbb{M}/2$ partial integrals. The solutions are obtained using the basic results of the first chapter. In general, the given set of first integrals is not necessarily in involution. The solution of the inverse problem in constrained Hamiltonian systems shows that in this case the differential equations having the invariant submanifold $\mathcal{M}$ are not in general Hamiltonian. The origin of the theory of noncommutative integration, dealing with Hamiltonian systems with first integrals that are not in involution, started with Nekhoroshev's Theorem.

Chapter 6 deals with the problem of *the integrability of a constrained rigid body*. We apply the results given in Chapter 4 to analyze the integrability of the motion of a rigid body around a fixed point. If the absence of constraints the integrability of this system is well known. But the integration of the equations of motion of this mechanical system with constraints is incomplete. We study two classical problems of constrained rigid bodies, the Suslov and the Veselova problems. We present new cases of integrability for these two problems which contain as particular cases the previously known results.

We also study the equations of motion of a constrained rigid body when the constraint is linear in the velocity with excluding the Lagrange multiplier. By using these equations we provide a simple proof of the well-known *theorem of Veselova* and improve *Kozlov's result* on the existence of an invariant measure. We give a new approach to solving the Suslov problem in the absence of a force field and of an invariant measure.

In Chapter 7 we give three main results:

(i) *A new point of view on transpositional relations.* In nonholonomic mechanics two points of view on transpositional relations have been maintained, one supported by Volterra, Hammel, and Hölder, and the other supported by Suslov, Voronets, and Levi-Civita. The second point of view has acquired general acceptance, while the first has been considered erroneous. We propose a third point of view, which is a generalization of the second one.

(ii) *A new generalization of the Hamiltonian principle.* There are two well-known generalizations of the Hamiltonian principle: the Hölder–Hamiltonian principle and the Suslov–Hamiltonian principle. We propose another generalization of the Hamiltonian principle, one that plays an important role in the solution of the inverse problem that we state in the next item.

(iii) *Statement and solution of the inverse problem in vakonomic mechanics.* We construct the variational equations of motion describing the behavior of constrained Lagrangian systems. Using the solution of the inverse problem in vakonomic mechanics, we present a modification of vakonomic mechanics (MVM). This modification is valid for holonomic and nonholonomic constrained Lagrangian systems. We deduce the equations of motion for nonholonomic systems with constraints that in general are nonlinear in the velocity. These equations coincide, except perhaps on a set of Lebesgue measure zero, with the classical differential equations deduced from the d'Alembert–Lagrange principle.

We observe that the solution of the inverse problem in vakonomic mechanics plays a fundamental role in the new point of view on transpositional relations and the new generalization of the Hamiltonian principle that we present.

Several aspects of our work support the following conjecture: *The existence of mechanical systems with constraints that are not linear in the velocity must be sought outside Newtonian Mechanics.*

Finally we remark that the inverse approach in ordinary differential equations which we propose in this book, based on the development of properties of the Nambu bracket, yields a unifed approach to the study of such diverse problems as finding all differential systems with given partial and first integrals, Hilbert's 16th problem, constrained Lagrangian and Hamiltonian systems, integrability of constrained rigid bodies, and vakonomic mechanics.

Contents

Chapter 1

Differential Equations with Given Partial and First Integrals

1.1 Introduction

In this chapter we present two different kind of results. First, under very general assumptions *we characterize the ordinary differential equations in* $\mathbb{R}^N$ *which have a given set of either* $M \leq N$, *or* $M > N$ *partial integrals, or* $M < N$ *first integrals, or* $M \leq N$ *partial and first integrals.* Second, in $\mathbb{R}^N$ we provide some results on integrability, in the sense that the characterized differential equations admit $N-1$ independent first integrals.

The main results of this chapter are proved by using the Nambu bracket. We establish new properties of this bracket.

For simplicity, we shall assume that all the functions which appear in this book are of class $\mathcal{C}^\infty$, although most of the results remain valid under weaker hypotheses.

The results obtained in this chapter are illustrated with concrete examples.

1.2 The Nambu bracket. New properties

In the 1970s, Nambu in [119] proposed a new approach to classical dynamics based on an N-dimensional Nambu–Poisson manifold replacing the even-dimensional Poisson manifold and on $N-1$ Hamiltonians $H_1, \ldots, H_{N-1}$ instead of a single Hamiltonian H. In the canonical Hamiltonian formulation, the equations of motion (Hamilton equations) are defined via the Poisson bracket. In Nambu's formulation, the Poisson bracket is replaced by the Nambu bracket. Nambu had originally considered the case $N = 3$.

Although the *Nambu formalism* is a generalization of the *Hamiltonian formalism*, its significant applications are not as rich as the applications of the latter.

Let D be an open subset of $\mathbb{R}^N$. Let $h_j = h_j(\mathbf{x})$ for $j = 1, 2, \ldots, M$ with $M \leq N$ be functions $h_j : \mathrm{D} \to \mathbb{R}$. We define the matrix

$$S_{M,N} = \begin{pmatrix} dh_1(\partial_1) & \ldots & dh_1(\partial_N) \\ \vdots & & \vdots \\ dh_M(\partial_1) & \ldots & dh_M(\partial_N) \end{pmatrix} = \begin{pmatrix} \partial_1 h_1 & \ldots & \partial_N h_1 \\ \vdots & & \vdots \\ \partial_1 h_M & \ldots & \partial_N h_M \end{pmatrix},$$

where $\partial_j h = \dfrac{\partial h}{\partial x_j}$ and $dh = \displaystyle\sum_{j=1}^{N} \partial_j h \, dx_j$. The matrix $S_{M,N}$ is also denoted by $\dfrac{\partial(h_1, \ldots, h_M)}{\partial(x_1, \ldots, x_N)}$.

We say that the functions h_j for $j = 1, \ldots, M \leq N$ are *independent* if the rank of the matrix $S_{M,N}$ is M for all $\mathbf{x} \in \mathrm{D}$, except perhaps in a subset of D of zero Lebesgue measure.

If $M = N$, we denote the matrix $S = S_{N,N}$. We note that S is the *Jacobian matrix* of the map $(h_1, \ldots, h_N)$. The *Jacobian* of S, i.e., the determinant of S, is denoted by

$$|S| = \left| \frac{\partial(h_1, \ldots, h_N)}{\partial(x_1, \ldots, x_N)} \right| = \begin{vmatrix} dh_1(\partial_1) & \ldots & dh_1(\partial_N) \\ \vdots & & \vdots \\ dh_N(\partial_1) & \ldots & dh_N(\partial_N) \end{vmatrix} := \{h_1, \ldots, h_N\}.$$

The last bracket thus defined is known in the literature as the *Nambu bracket* [7, 96, 119, 153].

The objective of this section is to establish a number of properties of the *Nambu bracket*, some of them new. These new properties will play an important role in some of the proofs of the main results of this book.

The Nambu bracket $\{h_1, \ldots, h_N\}$ has the following known properties.

(i) It is a *skew-symmetric* bracket, i.e.,

$$\{h_{\sigma(1)}, \ldots, h_{\sigma(N)}\} = (-1)^{|\sigma|}\{h_1, \ldots, h_N\}$$

for arbitrary functions $h_1, \ldots, h_N$ and arbitrary permutations σ of $(1, \ldots, N)$. Here $|\sigma|$ is the order of σ.

(ii) It is a derivation, i.e., satisfies the *Leibniz rule*

$$\{h_1, \ldots, fg\} = \{h_1, \ldots, f\}g + f\{h_1, \ldots, g\}.$$

(iii) It satisfies the *fundamental identity* (Filippov Identity)

$$\begin{aligned} &F(f_1 \ldots, f_{N-1}, g_1, \ldots, g_N) \\ &:= \{f_1 \ldots, f_{N-1}, \{g_1 \ldots, g_N\}\} \\ &\quad - \sum_{n=1}^{N} \{g_1, \ldots, g_{n-1}, \{f_1 \ldots, f_{N-1}, g_n\}, g_{n+1}, \ldots, g_N\} = 0, \end{aligned} \tag{1.1}$$

where $f_1, \ldots, f_{N-1}, g_1, \ldots, g_N$ are arbitrary functions. For more details see [96, 57, 119, 153]. (i) follows directly from the properties of determinants. (ii) is obtained using the properties of the derivative plus the properties of the determinants. (iii) will be the property (ix) with $\lambda = 1$, and we shall prove it in Proposition 1.2.2.

The properties listed above of the Nambu bracket are not sufficient for solving some of the problems which will arise in this book. The new properties that we give below will play a fundamental role in the proofs of some of the theorems and in the applications of the results in this book.

We emphasize that the applications of the Nambu bracket that we will give are original and represent a new direction developing Nambu's ideas.

We shall need the following results.

Proposition 1.2.1. *The following four identities hold.*

(iv) $$\sum_{j=1}^{N} \frac{\partial f}{\partial x_j}\{g_1, \ldots, g_{n-1}, x_j, g_{n+1}, \ldots, g_N\} = \{g_1, \ldots, g_{n-1}, f, g_{n+1} \ldots, g_N\}.$$

(v) $$\frac{\partial f}{\partial x_n} = \{x_1, \ldots, x_{n-1}, f, x_{n+1}, \ldots, x_N\}.$$

(vi) $$K_n^N := \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\{g_1, \ldots, g_{n-1}, x_j, g_{n+1} \ldots, g_N\} = 0, \text{ for } n = 1, 2, \ldots, N.$$

(vii) $$\frac{\partial f_1}{\partial x_N}\left|\frac{\partial (G, f_2, \ldots, f_N)}{\partial (y_1, \ldots, y_N)}\right| + \cdots + \frac{\partial f_N}{\partial x_N}\left|\frac{\partial (f_1, \ldots, f_{N-1}, G)}{\partial (y_1, \ldots, y_N)}\right| = \frac{\partial G}{\partial y_1}\left|\frac{\partial (f_1, \ldots, f_N)}{\partial (x_N, y_2, \ldots, y_N)}\right| + \cdots + \frac{\partial G}{\partial y_N}\left|\frac{\partial (f_1, \ldots, f_N)}{\partial (y_1, \ldots, y_{N-1}, x_N)}\right|.$$

Here the functions $g_1, \ldots, g_N, f_1, \ldots, f_N, G,$ and f are arbitrary.

Proof. The proof of (iv) reads

$$\{g_1, \ldots, g_{n-1}, f, g_{n+1} \ldots, g_N\} = \begin{vmatrix} \partial_1 g_1 & \ldots & \partial_N g_1 \\ \vdots & & \vdots \\ \partial_1 g_{n-1} & \ldots & \partial_N g_{n-1} \\ \partial_1 f & \ldots & \partial_N f \\ \partial_1 g_{n+1} & \ldots & \partial_N g_{n+1} \\ \vdots & & \vdots \\ \partial_1 g_N & \ldots & \partial_N g_N \end{vmatrix}$$

$$
= \partial_1 f \begin{vmatrix} \partial_1 g_1 & \partial_2 g_1 & \dots & \partial_N g_1 \\ \vdots & \vdots & & \vdots \\ \partial_1 g_{n-1} & \partial_2 g_{n-1} & \dots & \partial_N g_{n-1} \\ 1 & 0 & \dots & 0 \\ \partial_1 g_{n+1} & \partial_2 g_{n+1} & \dots & \partial_N g_{n+1} \\ \vdots & \vdots & & \vdots \\ \partial_1 g_N & \partial_2 g_N & \dots & \partial_N g_N \end{vmatrix} + \cdots
$$

$$
+ \partial_N f \begin{vmatrix} \partial_1 g_1 & \dots & \partial_{N-1} g_1 & \partial_N g_1 \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 g_{n-1} & \dots & \partial_{N-1} g_{n-1} & \partial_N g_{n-1} \\ 0 & \dots & 0 & 1 \\ \partial_1 g_{n+1} & \dots & \partial_{N-1} g_{n+1} & \partial_N g_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \partial_1 g_N & \dots & \partial_{N-1} g_N & \partial_N g_N \end{vmatrix}
$$

$$
\begin{aligned}
&= \{g_1, \dots, g_{n-1}, x_1, g_{n+1}, \dots, g_N\}\partial_1 f + \cdots \\
&\quad + \{g_1, \dots, g_{n-1}, x_N, g_{n+1}, \dots, g_N\}\partial_N f.
\end{aligned}
$$

The proof of (v) follows easily from the definition of the Nambu bracket.

The proof of (vi) is done by induction. Without loss of generality we shall prove that

$$
K_1^N = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\{x_j, g_2, \dots, g_N\} = 0. \tag{1.2}
$$

For $N = 2$ we have

$$
K_1^2 = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}\{x_j, g_2\} = \frac{\partial}{\partial x_1}\left(\frac{\partial g_2}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial g_2}{\partial x_1}\right) = 0.
$$

Now suppose that

$$
K_1^{N-1} = \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\{x_j, g_2, \dots, g_{N-1}\} = 0.
$$

We shall prove (1.2). Indeed, since

$$
\begin{aligned}
\{x_j, g_2, \dots, g_N\} &= \sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N}\{x_j, g_2, \dots, g_{k-1}, g_{k+1}, \dots, g_N\} \\
&\quad \text{for} \quad j = 1, \dots, N-1, \\
\{x_N, g_2, \dots, g_N\} &= (-1)^{N+1}\{g_2, \dots, g_N\},
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
K_1^N &= \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\{x_j, g_2, \ldots, g_N\} \\
&= \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\left(\sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N}\{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\right) \\
&\quad + \frac{\partial}{\partial x_N}\{x_N, g_2, \ldots, g_N\} \\
&= \sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\left(\frac{\partial g_k}{\partial x_N}\right)\{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\} \\
&\quad + \sum_{k=2}^{N} (-1)^{N+k+1} \frac{\partial g_k}{\partial x_N} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\left(\{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\}\right) \\
&\quad + (-1)^{N+1} \frac{\partial}{\partial x_N}\{g_2, \ldots, g_N\} \\
&= \sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\left(\frac{\partial g_k}{\partial x_N}\right)\{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\} \\
&\quad + (-1)^{N+1} \frac{\partial}{\partial x_N}\{g_2, \ldots, g_N\}.
\end{aligned}
$$

Here we used the inductive assumption that $K_1^{N-1} = 0$ with the functions $g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N$ instead of $g_2, \ldots, g_{k-1}, g_k, g_{k+1}, \ldots, g_{N-1}$.

In view of the property (iv) we obtain that

$$
\begin{aligned}
&\sum_{k=2}^{N} (-1)^{N+k+1} \sum_{j=1}^{N-1} \frac{\partial}{\partial x_j}\left(\frac{\partial g_k}{\partial x_N}\right)\{x_j, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\} \\
&\quad = \sum_{k=2}^{N} (-1)^{N} \left\{(-1)^{k+1} \frac{\partial g_k}{\partial x_N}, g_2, \ldots, g_{k-1}, g_{k+1}, \ldots, g_N\right\} \\
&\quad = \sum_{k=2}^{N} (-1)^{N} \left\{g_2, \ldots, g_{k-1}, \frac{\partial g_k}{\partial x_N}, g_{k+1}, \ldots, g_N\right\} \\
&\quad = (-1)^{N} \frac{\partial}{\partial x_N}\{g_2, \ldots, g_N\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
K_1^N &= \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\{x_j, g_2, \ldots, g_N\} \\
&= (-1)^{N} \frac{\partial}{\partial x_N}\{g_2, \ldots, g_N\} + (-1)^{N+1} \frac{\partial}{\partial x_N}\{g_2, \ldots, g_N\} = 0,
\end{aligned}
$$

and consequently the property (vi) is proved.

The proof of (vii) is easy to obtain by observing that the value of determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_N} & \frac{\partial f_1}{\partial x_N} \\ \vdots & \dots & \vdots & \vdots \\ \frac{\partial f_N}{\partial y_1} & \dots & \frac{\partial f_N}{\partial y_N} & \frac{\partial f_N}{\partial x_N} \\ \frac{\partial G}{\partial y_1} & \dots & \frac{\partial G}{\partial y_N} & 0 \end{vmatrix}$$

can be obtained by expanding by the last row and by the last column. □

Proposition 1.2.2. *We define*

$$\begin{aligned} &\Omega\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N, G\right) \\ &\quad := -\{f_1 \dots, f_{N-1}, G\}\{g_1 \dots, g_N\} \\ &\qquad + \sum_{n=1}^{N} \{f_1, \dots, f_{N-1}, g_n\}\{g_1, \dots, g_{n-1}, G, g_{n+1}, \dots, g_N\}, \end{aligned}$$

and

$$\begin{aligned} &F_\lambda\left(f_1 \dots, f_{N-1}, g_1, \dots, g_N\right) \\ &\quad := -\{f_1 \dots, f_{N-1}, \lambda\{g_1 \dots, g_N\}\} \\ &\qquad + \sum_{n=1}^{N} \{g_1, \dots, g_{n-1}, \lambda\{f_1 \dots, f_{N-1}, g_n\}, g_{n+1}, \dots, g_N\}, \end{aligned}$$

for arbitrary functions $f_1, \dots, f_{N-1}$, G, $g_1, \dots, g_N, \lambda$.

Then the Nambu bracket satisfies the identities:

(viii) $\Omega\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N, G\right) = 0$, *and*

(ix) $F_\lambda\left(f_1 \dots, f_{N-1}, g_1, \dots, g_N\right) = 0$. *Note this identity is a generalization of the fundamental identity* (1.1), *which is obtained when* $\lambda = 1$.

Proof. Indeed, $\Omega := \Omega\left(f_1 \dots, f_{N-1}, g_1 \dots, g_N, G\right)$ can be written as

$$\Omega = -\begin{vmatrix} dg_1(\partial_1) & \dots & dg_1(\partial_N) & \{f_1, \dots, f_{N-1}, g_1\} \\ \vdots & \dots & \vdots & \vdots \\ dg_N(\partial_1) & \dots & dg_N(\partial_N) & \{f_1, \dots, f_{N-1}, g_N\} \\ dG(\partial_1) & \dots & dG(\partial_N) & \{f_1, \dots, f_{N-1}, G\} \end{vmatrix},$$