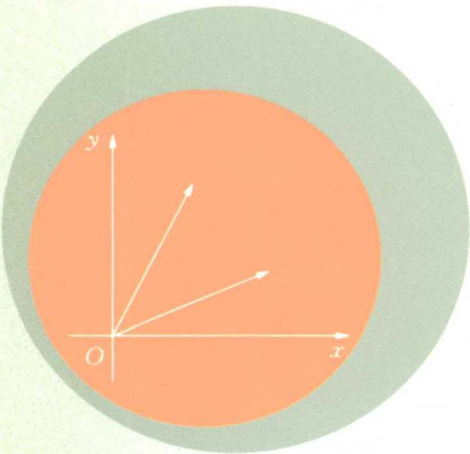
全国高等农林院校"十一五"规划教材

线性代数

英文版

LINEAR ALGEBRA

梁保松 曹殿立 主编



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Preface

This book is the 11th Five - Year programmed textbook for the countrywide higher agricultural university, and it is the fruit of teaching reform item of higher education in Henan Province.

At present age, the internationalization of higher education is the main trend of education development. Certainly, one of our primary goal in China's higher teaching revolution is to foster and train more and more highly special talents with international eyes, awareness, and association abilities. While the bilingual teaching is just a very good and effective way in realizing such a big goal.

In light of a document named "Suggestions on strengthening the higher undergraduate education and enhancing the quality in teaching", the ministry of education in 2001 proposed precisely that it is quite necessary in every common and specialized courses to provide conditions for undergraduate education in using English as a tool; further, it strove in three years to make the curriculum in foreign language covering 5%～10% among all courses given. In addition, the evaluating scheme for general higher undergraduate education issued by the ministry of education in 2002 brought the bilingual teaching an important check index into the assessment index system, requiring the hour for the bilingual teaching taking up more than 50% of a special course. Then, since the year 2002, the bilingual teaching has become one of the hottest topics in higher teaching revolution. Therefore, many universities propose policies encouraging teachers to study foreign teaching materials; accordingly, a new trying and exploration develop quickly in light of the practical course.

Since 2003, we've already made some exploration and tried on bilingual teaching, Creating an important factor, the environment for applying English, but not merely learn it. Also, such practice has been accepted by many students, from which we feel deep that teacher with good English are fundamental to the bilingual teaching; besides, to the foreign materials, only those contents satisfying our country's situation can be a crucial factor and would be more helpful.

Nowadays, the foreign texts introduced generally have common problems as follows:

1. The price is too high;

2. Many cases provided break away from China's social and life environment;

3. The contents and curriculum systems are quite different from that of our country.

Therefore, the problems referred above always impede the development of our country's bilingual teaching. In effect, by making native transformation, the foreign teaching materials can be applied much more effectively. Thus, in terms of the fostering characteristics for universities in our country combining with nowadays' curriculum contents and system, we complete the book "Linear Algebra" in English edition after making some digestion, assimilation, and recreation to the foreign text.

In publishing, I warmly thank the vice - headmaster Professor Baoan Cui, Henan Agricultural University, Mingzeng Yu, institution of higher education and Xiaoying Hu, for their highly support and help.

This edition of the Linear Algebra in English edition is a big try and exploration, and some mistakes may not be avoided despite of the most carefully arrangement and revision. So it will be our great pleasure to get criticism and comments from all experts, coteries and readers.

Baosong Liang
October 1, 2008

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Chapter 1 Determinants

Determinant is an important tool in mathematics research. This chapter will mainly introduce the properties of $n\times n$ determinants and Cramer's rule using determinant to solve linear system.

1.1 Determinants of Order 2 and 3

Use elimination to solve the system of linear equations in two unknowns

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases} \tag{1}$$

Where x_1 and x_2 are variables. To eliminate x_2, we multiply both sides of the two equations by a_{22} and a_{12}, respectively, and thus subtract the second equation from the first equation to obtain

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = a_{22}b_1 - a_{12}b_2;$$

Similarly, eliminate x_1 to obtain

$$(a_{11}a_{22} - a_{12}a_{21})x_2 = a_{11}b_2 - a_{21}b_1.$$

For $a_{11}a_{22}-a_{12}a_{21}\neq 0$, the solution of system (1) is given:

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}. \tag{2}$$

To express conveniently, the coefficients a_{11}, a_{12}, a_{21} and a_{22} of variables in (1) are arranged in two rows and two columns, and we introduce the symbol $D=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ to denote $a_{11}a_{22}-a_{12}a_{21}$, that is

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{3}$$

D is called a 2×2 **determinant.**

a_{11}, a_{12}, a_{21} and a_{22} are called **components** of the determinant, they are arranged in two rows and two columns. The horizontal line is called **row** while the vertical line is called **column.** And $a_{ij}\,(i, j=1, 2)$ is called the ij th component of determinant D.

The definition of the 2×2 determinant (3) can be remembered by using the **diagonal rule.** The line crossing a_{11} and a_{22} is called the **main diagonal**; The line crossing a_{21} and a_{12} is called the **subsidiary diagonal.** Thus 2×2 determinant is the product of two components in the main diagonal minus the product of two components in the subsidiary diagonal.

(3) is denoted as the denominator of the solution (2) to system (1) . According to the definition of 2×2 determinant, the numerators of the expressions of x_1 and x_2 in (2) are respec-

tively written as

$$D_1=\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}=a_{22}b_1-a_{12}b_2,\ D_2=\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}=b_2a_{11}-b_1a_{21}.$$

Obviously, $D_i(i=1, 2)$ is the determinant obtained by replacing the i th column of D by the constant column of (1) .

Thus if $D\neq 0$, the solution to the linear system (1) can be uniquely written as

$$x_1=\frac{D_1}{D},\ x_2=\frac{D_2}{D}. \tag{4}$$

This method is called **Cramer's Rule** used to solve the system of linear equations in two variables.

Example 1 Use 2×2 determinant to solve the system of linear equations

$$\begin{cases} 3x_1-2x_2=12, \\ 2x_1+\ x_2=1. \end{cases}$$

Solution Since $D=\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}=7\neq 0$ and

$$D_1=\begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix}=14,\ D_2=\begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix}=-21,$$

from (4) we obtain $$x_1=\frac{D_1}{D}=2,\ x_2=\frac{D_2}{D}=-3.$$

Consider the system of linear equations in three unknowns

$$\begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3=b_1, \\ a_{21}x_1+a_{22}x_2+a_{23}x_3=b_2, \\ a_{31}x_1+a_{32}x_2+a_{33}x_3=b_3. \end{cases} \tag{5}$$

Using the determinant method used to solve the system of linear equations in two unknowns, we arrange the coefficients of variables in system (5) in three rows and three columns and introduce the 3×3 **determinant**

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \tag{6}$$

D is called the coefficient determinant of system (5) . We replace the first , second, and third columns of D by the constant column of system (5) respectively, and introduce the following three determinants

$$D_1=\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},\ D_2=\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},\ D_3=\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix},$$

where D, D_1, D_2, D_3 are respectively defined as

$$D=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{13}a_{22}a_{31}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}, \tag{7}$$

$$D_1 = b_1a_{22}a_{33} + b_3a_{12}a_{23} + b_2a_{13}a_{32} - b_3a_{13}a_{22} - b_2a_{12}a_{33} - b_1a_{23}a_{32},$$
$$D_2 = b_2a_{11}a_{33} + b_1a_{23}a_{31} + b_3a_{13}a_{21} - b_2a_{13}a_{31} - b_1a_{21}a_{33} - b_3a_{11}a_{23},$$
$$D_3 = b_3a_{11}a_{22} + b_2a_{12}a_{31} + b_1a_{21}a_{32} - b_1a_{22}a_{31} - b_3a_{12}a_{21} - b_2a_{11}a_{32}.$$

If $D\neq 0$, the system (5) has an unique solution

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D},\ x_3 = \frac{D_3}{D}.$$

This method is called **Cramer's Rule** used to solve the system of linear equations in three variables.

The definition of 3×3 determinant can be remembered using the **diagonal rule.** It follows from (7) that we know D is constituted by 6 terms, each of which is the product of three components coming from distinct rows and columns of D. And the product is with sign, its rule is expressed as Figure 1 - 1. The three real lines in the figure are parallel to the main diagonal, and the product of three components in the real line is with positive sign; The three imaginary lines are parallel to the subsidiary diagonal, and the product of three components in the imaginary line is with negative sign.

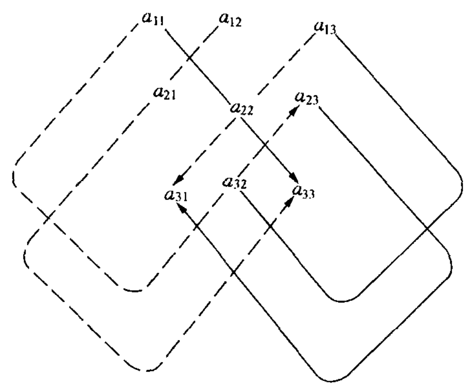

Figure 1 - 1

Example 2 Calculate

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 3 & 2 & 1 \\ -1 & 3 & -1 \end{vmatrix}.$$

Solution $D = 2\times 2\times(-1)+3\times 3\times 1+(-1)\times 1\times(-1)-(-1)$
$\times 2\times 1-3\times(-1)\times(-1)-2\times 1\times 3$
$=(-4)+9+1-(-2)-3-6=-1.$

1.2 Determinants of Order n

In upward chapter, we have used determinants of order 2 and 3 to introduce Cramer's Rule and solved the systems of linear equations in two and three unknowns. Cramer's Rule is valid for the system of n linear equations in n unknowns similarly, here we need to calculate $n\times n$ determinants. The diagonal rule used to calculate determinants of order 2 and 3 is not valid for determinants of order higher than 3, so we give the definition of $n\times n$ determinant and the general arithmetic.

It follows from (7) that we know:

(1) Each term of the expansion of determinant of order 3 is the product of three components

coming from distinct rows and columns;

(2) The expansion is composed of 3! terms, the row subscripts of three components of each term are arranged into the natural permutation while the column subscripts are arranged into a certain permutation of 1, 2, 3. The set of integers 1, 2, 3 has 3! distinct permutations and each permutation corresponds to one term of the expansion;

(3) The 3! terms of the expansion have three positive signs and three negative signs. The terms with positive sign correspond to the permutations (123), (312) and (231), respectively, which are obtained by zero or two (even number) times interchanges of any two numbers of the natural permutation 123; The permutations of column subscripts of the terms with negative sign are obtained by one (odd number) times interchanges of any two numbers of the natural permutation 123. That is, the sign of each term of the expansion relates to the number (even or odd) of interchanges.

To clarify the sign rule of the terms of expansion of $n\times n$ determinants, we introduce the definition of inversion.

1.2.1 Inversion and Odevity of Permutation

A permutation of order n is simply an arrangement of the n numbers 1, 2, $\cdots$, n into an ordered set without repetition, denoted by $i_1 i_2 \cdots i_n$. Obviously, there are $n!$ permutations of order n, and the permutation $12\cdots n$ is called the **natural permutation.**

Definition 1 In a permutation $i_1 i_2 \cdots i_n$, if a larger integer precedes a smaller one, then the two inintegers form an inversion. The total number of the inversions of a permutation is called the number of inversions, written $\tau(i_1 i_2 \cdots i_n)$.

A permutation is called **even permutation** if the number of inversions is even, and is called **odd permutation** if the number of inversions is odd.

Example 1 Find the number of inversions of the following permutation and determine whether the permutation is even or odd.

(1) 35214 ; (2) $n(n-1)\cdots 21$.

Solution According to the definition of the number of inversions, the number of inversions of any permutation $i_1 i_2 \cdots i_n$ can be written as

$\tau(i_1 i_2 \cdots i_n)$ = the number of integers that are less than i_1 and that follow i_1 + the number of integers that are less than i_2 and that follow $i_2 + \cdots +$ the number of integers that are less than i_{n-1} and that follow i_{n-1}.

(1) $\tau(35214)=2+3+1+0=6$, and the permutation 35214 is even;

(2) $\tau(n(n-1)\cdots 21) = (n-1)+(n-2)+\cdots+2+1=\frac{n(n-1)}{2}$.

The odevity of permutation $\frac{n(n-1)}{2}$ is determined by n and it is discussed as follows:

For $n=4k$, $\frac{n(n-1)}{2}=2k(4k-1)$ is even;

For $n=4k+1$, $\frac{n(n-1)}{2}=2k(4k+1)$ is even;

For $n=4k+2$, $\frac{n(n-1)}{2}=(2k+1)(4k+1)$ is odd;

For $n=4k+3$, $\frac{n(n-1)}{2}=(2k+1)(4k+3)$ is odd.

Thus for $n=4k$ or $n=4k+1$, the permutation is even; for $n=4k+2$ or $n=4k+3$, the permutation is odd.

Definition 2 In a permutation, if we interchange two integers i, j and other integers are fixed, we can obtain a new permutation. The transformation performed on the permutation is called a transposition, denoted by (i, j). The transposition of adjacent integers is called an adjacent transposition.

Theorem 1 A transposition changes the odevity of permutation.

Proof First we show that an adjacent transposition changes the odevity of permutation.

Interchanging the adjacent integers i, j in a permutation $\cdots ij \cdots$ of order n, we get a new permutation $\cdots ji \cdots$. Since integers other than i, j are fixed, the number of inversions of these integers are unchanged.

For $i>j$, the number of inversions decreases by 1; for $i<j$, the number of inversions increases by 1. Thus an adjacent transposition changes the odevity of permutation.

Secondly, we show that the general case is true.

Let there be a permutation $\cdots ia_1a_2\cdots a_kj\cdots$ of order n and k integers are between i and j. To interchange i, j and obtain a new permutation $\cdots ja_1a_2\cdots a_ki\cdots$, we can interchange i and a_1 and thus interchange i and a_2, ……Using $k+1$ times adjacent transpositions, we can move i to the position that j occupied and obtain a new permutation $\cdots a_1a_2\cdots a_kji\cdots$; and then we move j to the position that a_1 occupied by k times adjacent transpositions. In this way the interchange of i and j can be accomplished by $2k+1$ times adjacent transpositions. And the odevity of the original permutation is different from the new.

Corollary 1 The number of transpositions used to carry an odd permutation into the natural permutation is odd. The number of transpositions used to carry an even permutation into the natural permutation is even.

From Theorem 1, a transposition changes the odevity of a permutation. $12\cdots n$ is an even permutation, and if $i_1i_2\cdots i_n$ is an odd (even) permutation, then it must be carried into the natural permutation by transpositions an odd (even) number of times.

Corollary 2 In the set of permutations of order $n(n\geq 2)$, the number of odd permutations equals the number of even permutation and equals $\frac{n!}{2}$.

Proof Let there be p odd permutations and q even permutations in the set of permutations of order n. Using the same transposition (i, j) on the p odd permutations, it follows from Theorem 1 that we get p even permutations. If we use the transposition (i, j) on the p even permutations, we can also get the original p odd permutations, thus the p even permutations are different from each other. Since there are q even permutations together, $p\leqslant q$. Similarly, we can show that $q\leqslant p$, and thus $p=q$.

1.2.2 The Definition of Determinant of Order n

Used the definitions of the inversion of permutation and the odevity, the determinant of order 3

$$D=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

can be written in the form

$$\sum_{i_1 i_2 i_3} (-1)^{\tau(i_1 i_2 i_3)} a_{1i_1} a_{2i_2} a_{3i_3}.$$

The upward term can be generalized to the determinant of order n and we obtain

Definition 3 Symbol

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{1}$$

denotes a determinant of order n, it is an algebraic sum of $n!$ terms. These terms are all possible products $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$ of n components coming from distinct rows and columns of D. The sign of term $a_{1i_1} a_{2i_2} \cdots a_{ni_n}$ is $(-1)^{\tau(i_1 i_2 \cdots i_n)}$, if $i_1 i_2 \cdots i_n$ is an odd permutation, then the sign of the term is negative, if $i_1 i_2 \cdots i_n$ is an even permutation, then the sign of the term is positive. That is

$$D=\sum_{i_1 i_2 \cdots i_n} (-1)^{\tau(i_1 i_2 \cdots i_n)} a_{1i_1} a_{2i_2} \cdots a_{ni_n}. \tag{2}$$

Especially for $n=1$, $|a|=a$.

Example 2 Consider the determinant of order 4

$$D=\begin{vmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 0 & e & f & 0 \\ k & 0 & 0 & h \end{vmatrix},$$

according to the definition of determinant, D is an algebraic sum of $4!=24$ terms. Each term is a product of 4 components coming from distinct rows and columns of D. In the determinant, the terms other than $acfh$, $adeh$, $bdek$ and $bcfk$ comprise at least one factor 0, thus they equal 0. Permutations corresponding terms $acfh$, $adeh$, $bdek$ and $bcfk$, respectively, are 1 234, 1 324, 4 321 and 4 231, where the first and third are even permutations and the second and fourth are odd permutations. Thus we obtain

$$D = acfh - adeh + bdek - bcfk.$$

Example 3 Calculate the determinant of order n

$$D = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

All its components above the main diagonal are zero, so it is called **lower triangular** (all its components below the main diagonal are zero, so it is called **upper triangular**) .

Solution In $n!$ terms of the determinant D of order n, we consider the nonzero terms. Since each term is an algebraic product of n components coming from distinct rows and columns, the nonzero term must be a product of n nonzero components. In the first row only a_{11} is nonzero, so a_{1i_1} in (2) is only taken by a_{11}, and a_{2i_2} is taken by a_{22}, not a_{21}, since a_{21} and a_{11} are in the same column. Similarly a_{3i_3} is only taken by a_{33}, $\cdots$, the last is taken by a_{nn} and thus

$$D = (-1)^{\tau(12\cdots n)} a_{11}a_{22}\cdots a_{nn} = a_{11}a_{22}\cdots a_{nn}.$$

Similarly, the upper triangular

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

Especially, the determinant whose components off the main diagonal are zero (called **diagonal determinant**, written Λ)

$$\Lambda = \begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = a_{11}a_{22}\cdots a_{nn}.$$

In the same way, we can define **diagonal determinant** and **triangular determinant** corresponding to the subsidiary diagonal. Using the definition of determinant, as to diagonal determinant and upper、lower triangular determinant corresponding to the subsidiary diagonal, we, respectively, obtain the following conclusions:

$$\begin{vmatrix} & & & a_{1n} \\ & & a_{2n-1} & \\ & \iddots & & \\ a_{n1} & & & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2n-1} \cdots a_{n1};$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & \cdots & a_{2n-1} & \\ \vdots & \iddots & & \\ a_{n1} & & & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2n-1}\cdots a_{n1};$$

$$\begin{vmatrix} & & & a_{1n} \\ & & a_{2n-1} & a_{2n} \\ & \iddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2n-1}\cdots a_{n1}.$$

It should be pointed out that the definition of determinant of order n has many forms.

For example, let the column subscripts of each term of determinant of order n be arranged into the natural permutation, whereas the row subscripts form a certain permutation $j_1j_2\cdots j_n$ of order n. Thus we obtain another form of the definition of determinant (1)

$$D = \sum_{j_1j_2\cdots j_n} (-1)^{\tau(j_1j_2\cdots j_n)} a_{j_1 1}a_{j_2 2}\cdots a_{j_n n}. \tag{3}$$

If the row subscripts of determinant of order n form a permutation $i_1i_2\cdots i_n$ of order n, then the column subscripts form another permutation $j_1j_2\cdots j_n$ of order n, then determinant (1) can also be defined as the following form

$$D = \sum (-1)^{\tau(i_1i_2\cdots i_n)+\tau(j_1j_2\cdots j_n)} a_{i_1j_1}a_{i_2j_2}\cdots a_{i_nj_n}. \tag{4}$$

1.3 Properties of Determinant

It follows from Definition 3 that we need to calculate $n!$ products using the definition of determinant to calculate the determinant of order n, it is very inconvenient. Thus we will study properties of determinant in the following way and use the properties to simplify the calculations.

First we introduce the definition of the transpose of determinant.

Consider the determinant of order n

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

Interchanging the rows and columns of D, we obtain a new determinant

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix},$$