

Graduate Texts in Mathematics

John M. Lee

Introduction to Topological Manifolds 拓扑流形引论

Springer-Verlag

世界图书出版公司

John M. Lee

Introduction to Topological Manifolds

With 138 Illustrations



Springer

书 名: Introduction to Topological Manifolds
作 者: J. M. Lee
中 译 名: 拓扑流形引论
出 版 者: 世界图书出版公司北京公司
印 刷 者: 北京世图印刷厂
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)
联系电话: 010-64015659, 64038347
电子信箱: kjsk@vip.sina.com
开 本: 24 印 张: 17
出版年代: 2003 年 6 月
书 号: 7-5062-5959-1/O · 378
版权登记: 图字: 01-2003-3603
定 价: 38.00 元

世界图书出版公司北京公司已获得 Springer-Verlag 授权在中国
大陆独家重印发行。

Graduate Texts in Mathematics 202

Editorial Board

S. Axler F.W. Gehring K.A. Ribet

Springer

New York

Berlin

Heidelberg

Barcelona

Hong Kong

London

Milan

Paris

Singapore

Tokyo

Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
- 2 OXToby. Measure and Category. 2nd ed.
- 3 SCHAEFER. Topological Vector Spaces. 2nd ed.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra. 2nd ed.
- 5 MAC LANE. Categories for the Working Mathematician. 2nd ed.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable I. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
- 14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
- 20 HUSEMOLLER. Fibre Bundles. 3rd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and Its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 ALEXANDER/WERMER. Several Complex Variables and Banach Algebras. 3rd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to C^* -Algebras.
- 40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOÈVE. Probability Theory I. 4th ed.
- 46 LOÈVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.
- 48 SACHS/WU. General Relativity for Mathematicians.
- 49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
- 50 EDWARDS. Fermat's Last Theorem.
- 51 KLINGENBERG. A Course in Differential Geometry.
- 52 HARTSHORNE. Algebraic Geometry.
- 53 MANIN. A Course in Mathematical Logic.
- 54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
- 55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
- 56 MASSEY. Algebraic Topology: An Introduction.
- 57 CROWELL/FOX. Introduction to Knot Theory.
- 58 KOBLITZ. p -adic Numbers, p -adic Analysis, and Zeta-Functions. 2nd ed.
- 59 LANG. Cyclotomic Fields.
- 60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.
- 61 WHITEHEAD. Elements of Homotopy
- 62 KARGAPOLOV/MERLJAKOV. Fundamentals of the Theory of Groups.
- 63 BOLLOBAS. Graph Theory.
- 64 EDWARDS. Fourier Series. Vol. I 2nd ed.
- 65 WELLS. Differential Analysis on Complex Manifolds. 2nd ed.
- 66 WATERHOUSE. Introduction to Affine Group Schemes.
- 67 SERRE. Local Fields.
- 68 WEIDMANN. Linear Operators in Hilbert Spaces.
- 69 LANG. Cyclotomic Fields II.
- 70 MASSEY. Singular Homology Theory.
- 71 FARKAS/KRA. Riemann Surfaces. 2nd ed.

(continued after index)

John M. Lee
Department of Mathematics
University of Washington
Seattle, WA 98195-4350
USA
lee@math.washington.edu

Editorial Board

S. Axler
Mathematics Department
San Francisco State
University
San Francisco, CA 94132
USA

F.W. Gehring
Mathematics Department
East Hall
University of Michigan
Ann Arbor, MI 48109
USA

K.A. Ribet
Mathematics Department
University of California
at Berkeley
Berkeley, CA 94720-3840
USA

Mathematics Subject Classification (2000): 54-01, 55-01, 57-01, 57N65

Library of Congress Cataloging-in-Publication Data

Lee, John M., 1950–

Introduction to topological manifolds / John M. Lee.

p. cm. — (Graduate texts in mathematics ; 202)

Includes bibliographical references and index.

ISBN 0-387-98759-2 (hard cover : alk. paper) — ISBN 0-387-95026-5 (soft cover : alk. paper)

1. Topological manifolds. I. Title. II. Series.

QA613.2 .L44 2000

514'.3—dc21

00-026156

Printed on acid-free paper.

© 2000 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden. The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.

Reprinted in China by Beijing World Publishing Corporation, 2003

9 8 7 6 5 4 3 2 1

ISBN 0-387-98759-2 Springer-Verlag New York Berlin Heidelberg SPIN 10711695(hardcover)
ISBN 0-387-95026-5 Springer-Verlag New York Berlin Heidelberg SPIN 10763406 (softcover)

*To Pm,
sine qua non*

Preface

This book is an introduction to manifolds at the beginning graduate level. It contains the essential topological ideas that are needed for the further study of manifolds, particularly in the context of differential geometry, algebraic topology, and related fields. Its guiding philosophy is to develop these ideas rigorously but economically, with minimal prerequisites and plenty of geometric intuition. Here at the University of Washington, for example, this text is used for the first third of a year-long course on the geometry and topology of manifolds; the remaining two-thirds focuses on smooth manifolds.

There are many superb texts on general and algebraic topology available. Why add another one to the catalog? The answer lies in my particular vision of graduate education—it is my (admittedly biased) belief that every serious student of mathematics needs to know manifolds intimately, in the same way that most students come to know the integers, the real numbers, Euclidean spaces, groups, rings, and fields. Manifolds play a role in nearly every major branch of mathematics (as I illustrate in Chapter 1), and specialists in many fields find themselves using concepts and terminology from topology and manifold theory on a daily basis. Manifolds are thus part of the basic vocabulary of mathematics, and need to be part of the basic graduate education. The first steps must be topological, and are embodied in this book; in most cases, they should be complemented by material on smooth manifolds, vector fields, differential forms, and the like. (After all, few of the really interesting applications of manifold theory are possible without using tools from calculus.)

MPG 54 / 06

Of course, it is not realistic to expect all graduate students to take full-year courses in general topology, algebraic topology, and differential geometry. Thus, although this book touches on a generous portion of the material that is typically included in much longer courses, the coverage is selective and relatively concise, so that most of the book can be covered in a single quarter or semester, leaving time in a year-long course for further study in whatever direction best suits the instructor and the students. At U.W. we follow it with a two-quarter sequence on smooth manifold theory; but it could equally well lead into a full-blown course on algebraic topology.

It is easy to describe what this book is not. It is not a course on general topology—many of the topics that are standard in such a course are ignored here, such as metrization theorems; infinite products and the Tychonoff theorem; countability and separation axioms and the relationships among them (other than second countability and the Hausdorff axiom, which are part of the definition of manifolds); and function spaces. Nor is it a course in algebraic topology—although I treat the fundamental group in detail, there is barely a mention of the higher homotopy groups, and the treatment of homology theory is extremely brief, meant mainly to give the flavor of the theory and to lay some groundwork for the later introduction of de Rham cohomology. It is certainly not a comprehensive course on topological manifolds, which would have to include such topics as PL structures and maps, transversality, intersection theory, cobordism, bundles, characteristic classes, and low-dimensional geometric topology. Finally, it is not intended as a reference book, because few of the results are presented in their most general or most complete form.

Perhaps the best way to summarize what this book is would be to say that it represents, to a good approximation, my conception of the ideal amount of topological knowledge that should be possessed by beginning graduate students who are planning to go on to study smooth manifolds and differential geometry. Experienced mathematicians will probably observe that my choices of material and approach have been influenced by the fact that I am a differential geometer and analyst by training and predilection, not a topologist. Thus I give special emphasis to topics that will be of importance later in the study of smooth manifolds, such as group actions, orientations, and degree theory. (A few topological ideas that are important for manifold theory, such as paracompactness and embedding theorems, are omitted because they are better treated in the context of smooth manifolds.) But despite my prejudices, I have tried to make the book useful as a precursor to algebraic topology courses as well, and it could easily serve as a prerequisite to a more extensive course in homology and homotopy theory.

Prerequisites. The prerequisite for studying this book is, briefly stated, a solid undergraduate degree in mathematics; but this probably deserves some elaboration. Traditionally, “algebraic topology” has been seen as a

separate subject from “general topology,” and most courses in the former begin with the assumption that the students have already completed a course in the latter. However, the sad fact is that for a variety of reasons, many undergraduate mathematics majors in the U.S. never take a course in general topology. For that reason I have written this book without assuming that the reader has had any exposure to topological spaces. On the other hand, I do assume several essential prerequisites beyond calculus and linear algebra: basic logic and set theory such as what one would encounter in any rigorous undergraduate analysis or algebra course; real analysis at the level of Rudin’s *Principles of Mathematical Analysis* [Rud76], including, in particular, a thorough understanding of metric spaces and their continuous functions and compact subsets; and group theory at the level of Hungerford’s *Abstract Algebra: An Introduction* [Hun90] or Herstein’s *Topics in Algebra* [Her75]. Because it is vitally important that the reader be comfortable with this prerequisite material, I have collected in the Appendix a summary of the main points that are used throughout the book, together with a representative collection of exercises. These exercises, which should be relatively straightforward for anyone who has had the prerequisite courses, can be used by the student to refresh his or her knowledge, or can be assigned by the instructor at the beginning of the course to make sure that everyone starts with the same background.

Organization. The book is divided into thirteen chapters, which can be grouped into an introduction and five major substantive sections.

The introduction (Chapter 1) is meant to whet the student’s appetite and create a “big picture” into which the many details can later fit.

The first major section, Chapters 2 through 4, is a brief and highly selective introduction to the ideas of general topology: topological spaces; their subspaces, products, and quotients; and connectedness and compactness. Of course, manifolds are the main examples and are emphasized throughout. These chapters emphasize the ways in which topological spaces differ from the more familiar Euclidean and metric spaces, and carefully develop the machinery that will be needed later, such as quotient maps, local path connectedness, and locally compact Hausdorff spaces.

The second major section, comprising Chapters 5 and 6, explores in detail the main examples that motivate the rest of the theory: simplicial complexes, 1-manifolds, and 2-manifolds. Chapter 5 introduces simplicial complexes in two ways—first concretely, as locally finite collections of simplices in Euclidean space that intersect nicely; and then abstractly, as collections of finite vertex sets. Both approaches are useful: The concrete definition helps students develop their geometric intuition, while the abstract point of view emphasizes the fact that all statements about simplicial complexes can be reduced to combinatorics. There are several reasons for introducing simplicial complexes at this stage: They furnish a rich source of examples; they give a very concrete way of thinking about orientations and the Euler

characteristic; they provide the concept of triangulability needed for the classifications of 1-manifolds and 2-manifolds; and they set the stage for the treatment of homology later. Chapter 6 begins by proving a classification theorem for 1-manifolds using the triangulability theorem proved in the preceding chapter. The rest of the chapter is devoted to a detailed study of 2-manifolds. After exploring the basic examples of surfaces—the sphere, the torus, the projective plane, and their connected sums—I give a complete proof of the classification theorem for compact surfaces, essentially following the treatment in [Mas89].

The third major section, Chapters 7 through 10, is the core of the book. In it, I give a fairly complete and traditional treatment of the fundamental group. Chapter 7 introduces the definitions and proves the topological and homotopy invariance of the fundamental group. At the end of the chapter I insert a brief introduction to category theory. Categories are not used in a central way anywhere in the book, but it is natural to introduce them after having proved the topological invariance of the fundamental group, and it is useful for students to begin thinking in categorical terms early. Chapter 8 gives a detailed proof that the fundamental group of the circle is infinite cyclic. Because the techniques used here are the precursor and motivation for the entire theory of covering spaces, I introduce some of the terminology of the latter subject—evenly covered neighborhoods, local sections, lifting—in the special case of the circle, and the proofs here form a model for the proofs of more general theorems involving covering spaces to come in a later chapter. Chapter 9 is a brief digression into group theory. Although a basic acquaintance with group theory is an essential prerequisite, most undergraduate algebra courses do not treat free products, free groups, presentations of groups, or free abelian groups, so I develop these subjects from scratch. (The material on free abelian groups is included primarily for use in the treatment of homology in Chapter 13, but some of the results play a role also in classifying the coverings of the torus in Chapter 12.) The last chapter of this section gives the statement and proof of the Seifert–Van Kampen theorem, which expresses the fundamental group of a space in terms of the fundamental groups of its subsets, and describes several applications of the theorem including computation of the fundamental groups of graphs and of all the compact surfaces.

The fourth major section consists of two chapters on covering spaces. Chapter 11 defines covering spaces, gives a few examples, and develops the theory of the covering group. Much of the development goes rapidly here, because it is parallel to what was done earlier in the concrete case of the circle. The ostensible goal of Chapter 12 is to prove the classification theorem for coverings—that there is a one-to-one correspondence between isomorphism classes of coverings of X and conjugacy classes of subgroups of the fundamental group of X —but along the way two other ideas are developed that are of central importance in their own right. The first is the notion of the universal covering space, together with proofs that every manifold has a

universal covering and that the universal covering space covers every other covering space. The second is the fact that the quotient of a manifold by a free, proper action of a discrete group yields a manifold. These ideas are applied to a number of important examples, including classifying coverings of the torus and the lens spaces, and proving that surfaces of higher genus are covered by the hyperbolic disk.

The fifth major section of the book consists of one chapter only, Chapter 13, on homology theory. In order to cover some of the most important applications of homology to manifolds in a reasonable time, I have chosen a "low-tech" approach to the subject. I focus mainly on singular homology because it is the most straightforward generalization of the fundamental group. After defining the homology groups, I prove a few essential properties, including homotopy invariance and the Mayer-Vietoris theorem, with a minimum of homological machinery. I could not resist including a (terribly brief) introduction to simplicial homology, just because it immediately yields the topological invariance of the Euler characteristic. The last section of the chapter is a brief introduction to cohomology, mainly with field coefficients, to serve as background for a treatment of de Rham theory in a later course. In keeping with the overall philosophy of focusing only on what is necessary for a basic understanding of manifolds, I do not even mention relative homology, homology with arbitrary coefficients, simplicial approximation, or the axioms for a homology theory.

Although this book grew out of notes designed for a one-quarter graduate course, there is clearly too much material here to cover adequately in ten weeks. It should be possible to cover all or most of it in a semester with well prepared students. The book could even be used for a full-year course, allowing the instructor to adopt a much more leisurely pace and to work out some of the problems as examples in class.

Each instructor will have his or her own ideas about what to leave out in order to fit the material into a short course. At the University of Washington, we typically do not cover the chapter on homology at all, and give short shrift to some of the simplicial theory and some of the more involved examples of covering maps. Others may wish to leave out some or all of the material on covering spaces, or the classification of surfaces. With students who have had a solid topology course, the first four chapters could be skipped or assigned as outside reading.

Exercises and Problems. As is the case with any new mathematical material, and perhaps even more than usual with material like this that is so different from the mathematics most students have seen as undergraduates, it is impossible to learn the subject without getting one's hands dirty and working out a large number of examples and problems. I have tried to give the reader ample opportunity to do so throughout the book. In every chapter, and especially in the early ones, there are "exercises" woven into the text. Do not ignore them; without their solutions, the text is incomplete.

The reader should take each exercise as a signal to stop reading, pull out a pencil and paper, and work out the answer before proceeding further. The exercises are usually relatively easy, and typically involve proving minor results or working out examples that are essential to the flow of the exposition. Some require techniques that the student probably already knows from prior courses; others ask the student to practice techniques or apply results that have recently been introduced in the text. A few are straightforward but rather long arguments that are more enlightening to work through on one's own than to read. In the later chapters, fewer things are singled out as exercises, but there are still plenty of omitted details in the text that the student should work out before going on; it is my hope that by the time the student reaches the last few chapters he or she will have developed the habit of stopping and working through most of the details that are not spelled out without having to be told.

At the end of each chapter is a selection of "problems." These are, with a few exceptions, harder and/or longer than the exercises, and give the student a chance to grapple with more significant issues. The results of a number of the problems are used later in the text. There are more problems than most students could do in a quarter or a semester, so the instructor will want to decide which ones are most germane and assign those as homework.

Acknowledgments. Those of my colleagues at the University of Washington with whom I have discussed this material—Tom Duchamp, Judith Arms, Steve Mitchell, Scott Osborne, and Ethan Devinatz—have provided invaluable help in sorting out what should go into this book and how it should be presented. Each has had a strong influence on the way the book has come out, for which I am deeply grateful. (On the other hand, it is likely that none of them would wholeheartedly endorse all my choices regarding which topics to treat and how to treat them, so they are not to be blamed for any awkwardnesses that remain.) I would like to thank Ethan Devinatz in particular for having had the courage to use the book as a course text when it was still in an inchoate state, and for having the grace and patience to wait while I prepared chapters at the last minute for his course.

Thanks are due also to Mary Sheetz, who did an excellent job producing some of the illustrations under the pressures of time and a finicky author.

My debt to the authors of several other textbooks will be obvious to anyone who knows those books: William Massey's *Algebraic Topology: An Introduction* [Mas89], Allan Sieradski's *An Introduction to Topology and Homotopy* [Sie92], Glen Bredon's *Topology and Geometry*, and James Munkres's *Topology: A First Course* [Mun75] and *Elements of Algebraic Topology* [Mun84] are foremost among them.

Finally, I would like to thank my wife, Pm, for her forbearance and ~~unflagging~~ support while I was spending far too much time with this book

and far too little with the family; without her help I unquestionably could not have done it.

John M. Lee

Seattle

Contents

Preface	vii
1 Introduction	1
What Are Manifolds?	1
Why Study Manifolds?	4
2 Topological Spaces	17
Topologies	17
Bases	27
Manifolds	30
Problems	36
3 New Spaces from Old	39
Subspaces	39
Product Spaces	48
Quotient Spaces	52
Group Actions	58
Problems	62
4 Connectedness and Compactness	65
Connectedness	65
Compactness	73
Locally Compact Hausdorff Spaces	81
Problems	88

5	Simplicial Complexes	91
	Euclidean Simplicial Complexes	92
	Abstract Simplicial Complexes	96
	Triangulation Theorems	102
	Orientations	105
	Combinatorial Invariants	109
	Problems	114
6	Curves and Surfaces	117
	Classification of Curves	118
	Surfaces	119
	Connected Sums	126
	Polygonal Presentations of Surfaces	129
	Classification of Surface Presentations	137
	Combinatorial Invariants	142
	Problems	146
7	Homotopy and the Fundamental Group	147
	Homotopy	148
	The Fundamental Group	150
	Homomorphisms Induced by Continuous Maps	158
	Homotopy Equivalence	161
	Higher Homotopy Groups	169
	Categories and Functors	170
	Problems	176
8	Circles and Spheres	179
	The Fundamental Group of the Circle	180
	Proofs of the Lifting Lemmas	183
	Fundamental Groups of Spheres	187
	Fundamental Groups of Product Spaces	188
	Fundamental Groups of Manifolds	189
	Problems	191
9	Some Group Theory	193
	Free Products	193
	Free Groups	199
	Presentations of Groups	201
	Free Abelian Groups	203
	Problems	208
10	The Seifert–Van Kampen Theorem	209
	Statement of the Theorem	210
	Applications	212
	Proof of the Theorem	221

Distinguishing Manifolds	227
Problems	230
11 Covering Spaces	233
Definitions and Basic Properties	234
Covering Maps and the Fundamental Group	239
The Covering Group	247
Problems	253
12 Classification of Coverings	257
Covering Homomorphisms	258
The Universal Covering Space	261
Proper Group Actions	266
The Classification Theorem	283
Problems	289
13 Homology	291
Singular Homology Groups	292
Homotopy Invariance	300
Homology and the Fundamental Group	304
The Mayer–Vietoris Theorem	308
Applications	318
The Homology of a Simplicial Complex	323
Cohomology	329
Problems	334
Appendix: Review of Prerequisites	337
Set Theory	337
Metric Spaces	347
Group Theory	352
References	359
Index	362