COMBINATORIAL REASONING

An Introduction to the Art of Counting

DUANE DeTEMPLE
WILLIAM WEBB

WILEY

COMBINATORIAL REASONING

An Introduction to the Art of Counting

DUANE DeTEMPLE
WILLIAM WEBB
Department of Mathematics
Washington State University
Pullman, WA

WILEY

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

DeTemple, Duane W.
Combinatorial reasoning : an introduction to the art of counting / Duane DeTemple, William Webb.
pages cm
Includes bibliographical references and index.
ISBN 978-1-118-65218-3 (cloth)
1. Combinatorial analysis–Textbooks. 2. Mathematical analysis–Textbooks. I. Webb, William, 1944– II. Title.
QA164.D48 2013
511′.6–dc23

2013035522

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

COMBINATORIAL REASONING

PREFACE

Counting problems, or more precisely enumerative combinatorics, are a source of some of the most intriguing problems in mathematics. Often these problems can be solved using ingenious and creative observations, what we call *combinatorial reasoning*. It is this kind of thinking that we stress throughout the descriptions, examples, and problems in this text.

Combinatorics has many important applications to areas as diverse as computer science, probability and statistics, and discrete optimization. But equally important, the subject offers many results of beautiful mathematics that are enjoyable to discover and problems that are simply fun to think about and solve in innovative ways.

Each of us has over 40 years of experience teaching combinatorics, as well as other mathematics courses, at both the undergraduate and graduate levels. We think that we have learned some effective ways to present this subject. Early versions of the notes for the book were used in both undergraduate and graduate courses, and our students found the approach both easy to understand and quite thorough.

FEATURES OF THIS TEXT

Chapter 1 introduces the reader to combinatorial thinking by considering topics of existence, construction, and enumeration that lead, by the end of the chapter, to general principles of combinatorics that are employed throughout the remainder of the text. The problems solved in this chapter, often involving dot patterns and tilings of rectangular boards, are easily described and visualized, and foreshadow much of what comes later.

More formal considerations of combinatorics are taken up in Chapter 2, where selections, arrangements, and distribution are treated in detail. Special attention is given to combinatorial models—block walking, tiling of rectangular boards, committee selection, and others. It is shown how general results can be derived by combinatorial reasoning based on an appropriate model. Most often, only the simplest of algebraic calculations are necessary.

An unusually complete discussion of generating functions—both ordinary and exponential—is given in Chapter 3, where the binomial series is shown to be a prototype of a much larger collection of generating functions. It is also shown how enumeration problems can often be solved with generating functions.

Chapter 4 begins with the DIE method (Description—Involution—Exceptions), which is shown to be a powerful combinatorial approach to the evaluation of alternating series. This leads naturally to the Principle of Inclusion/Exclusion. The chapter then turns to a section on rook polynomials that combines generating functions and inclusion–exclusion to solve an interesting class of restricted arrangement problems. The chapter concludes with an optional section on the Zeckendorf representation of integers and its application to creating a winning strategy for the game Fibonacci Nim.

Recurrence sequences are treated in detail in Chapters 5, using what we call the operator method. By employing the readily understood successor operator E, which simply replaces n by $n + 1$, the properties of recurrence sequences seem natural and easy to understand. This approach not only deepens comprehension but also simplifies many calculations.

Chapter 6 enlarges the library of special numbers that often answer combinatorial questions. Since the sections within this chapter are largely independent, there is freedom to pursue whatever topics seem of most interest—Stirling numbers, harmonic numbers, Bernoulli numbers, Eulerian numbers, partition numbers, or Catalan numbers.

Chapter 7 returns to the operator approach for solving linear recurrence relations that was introduced earlier in Chapter 5. Here, by viewing recurrence sequences as vector spaces, additional methods to solve recurrence relations become available. Moreover, we discover a powerful new approach to both discover and verify combinatorial identities.

Pólya-Redfield counting—the enumeration of arrangements that take symmetries into account—is the subject of Chapter 8, the final chapter. Here, abstraction is minimized by showing how general formulas can be derived from the consideration of carefully chosen simple figures and arrangements.

FLEXIBILITY FOR COURSES

A beginning course for undergraduates can be easily constructed using selected sections from Chapters 1–5. A course for advanced undergraduates and beginning graduate students might give quicker coverage to early chapters and include material chosen from Chapters 6, 7, and 8. For example, Sections 6.1–6.6, on special numbers, are largely independent, and any of these sections can be covered in any order. There are no special prerequisites for this material beyond a little exposure to power series. An elementary introduction to linear algebra is needed for Chapter 7. A little background in group theory is helpful for Chapter 8, but in our experience this chapter can provide a good introduction to this algebraic topic.

We would like to thank Ken Davis (Hardin-Simmons University) for his very helpful suggestions and comments.

DUANE DETEMPLE
WILLIAM WEBB

Washington State University
Pullman

CONTENTS

PART II TWO ADDITIONAL TOPICS IN ENUMERATION

PART III NOTATIONS INDEX, APPENDICES, AND SOLUTIONS TO SELECTED ODD PROBLEMS

PART I

THE BASICS OF ENUMERATIVE COMBINATORICS

1

INITIAL EnCOUNTers WITH COMBINATORIAL REASONING

1.1 INTRODUCTION

Although this text is devoted largely to enumerative combinatorics, Section 1.2 presents a brief encounter with a simple yet surprisingly versatile method to prove existence, the pigeonhole principle. Section 1.3 discusses some combinatorial construction problems associated with covering a chessboard with dominoes. In Section 1.4, we consider some number sequences that often arise in combinatorial problems such as *triangular* numbers 1,3,6,10, . . .; *square* numbers 1,4,9,16, . . .; and other *figurate numbers*, where the terminology alludes to the representation of these numbers by geometric patterns of dots. In Section 1.5, we count the number of ways a $1 \times n$ rectangle can be tiled with either unit squares of two contrasting colors or with a mixture of 1×1 squares and 1×2 dominoes. By counting the number of dots in a pattern or the number of tilings of a chessboard, we will discover several general principles of counting that are fundamental to enumerative combinatorics. In particular, we will encounter the addition and multiplication principles, which are explored in detail in Section 1.6, which concludes the chapter.

1.2 THE PIGEONHOLE PRINCIPLE

The pigeonhole principle was first applied in 1834 by Peter Dirichlet (1805–1859) to solve a problem in number theory. Soon, other mathematicians found his idea equally

Combinatorial Reasoning: An Introduction to the Art of Counting, First Edition. Duane DeTemple and William Webb.

useful and referred to it as *Dirichlet's box principle* (*Schubfachprinzip* in German). Later, in the nineteenth century, the term *pigeonhole* was used in reference to the small boxes or drawers common in desks of that century. (It may be comforting to know that envelopes, and not pigeons, are placed in the pigeonholes.)

Dirichlet's idea is simply stated as follows.

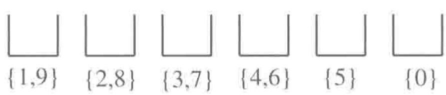
Theorem 1.1 (Pigeonhole Principle) If $n + 1$ or more objects are placed into n boxes, then at least one of the boxes contains two or more of the objects.

Proof (by contradiction). Suppose, to the contrary, that each of the n boxes contains no more than one object. Then the n boxes together contain no more than n objects, a contradiction. ■

The examples that follow show how the pigeonhole principle provides the basis for an existence proof. It is helpful to think of pigeons and pigeonholes as metaphorical terms for the objects and boxes of the theorem.

Example 1.2 In a family of seven, there must be two family members for which either the sum or difference in their ages can be given in decades, that is, as a multiple of 10.

Solution. A multiple of 10 is easy to identify by the digit 0 in the units position. If two family members have ages ending with the same digit, the difference in their ages is a multiple of 10. Also, if someone in the family has an age ending in the digit 1 and another family member's age ends with the digit 9, their sum of ages ends with the digit 0. Continuing with this type of reasoning suggests that we define the following pigeonholes:



The *pigeons* are the seven ages of the family member. When these are placed in the box labeled with the set containing the last digit of the age, the pigeonhole principle guarantees that at least one of the six boxes contains at least two people with one of the labeled ages. If these two ages happen to have the same unit's digit, then their difference is a multiple of 10. If the two ages have different last digits, these must be 1 and 9, or 2 and 8, or 3 and 7, or 4 and 6. In each case, the sum of the two ages is a multiple of 10. ■

The pigeonholes set up for Example 1.2 show why seven numbers were needed. There is no pair of numbers from the six members of the set $\{1, 2, 3, 4, 5, 10\}$ whose sum or difference is divisible by 10.