CONTEMPORARY MATHEMATICS

599

Geometric Analysis, Mathematical Relativity, and Nonlinear Partial Differential Equations

Southeast Geometry Seminars
Emory University, Georgia Institute of Technology,
University of Alabama, Birmingham,
and the University of Tennessee
2009–2011

Mohammad Ghomi
Junfang Li
John McCuan
Vladimir Oliker
Fernando Schwartz
Gilbert Weinstein
Editors

American Mathematical Society

This volume presents the proceedings of the Southeast Geometry Seminar for the meetings that took place bi-annually between the fall of 2009 and the fall of 2011, at Emory University, Georgia Institute of Technology, University of Alabama Birmingham, and the University of Tennessee. Talks at the seminar are devoted to various aspects of geometric analysis and related fields, in particular, nonlinear partial differential equations, general relativity, and geometric topology.

Articles in this volume cover the following topics: a new set of axioms for General Relativity, CR manifolds, the Mañé Conjecture, minimal surfaces, maximal measures, pendant drops, the Funk-Radon-Helgason method, ADM-mass and capacity, and extrinsic curvature in metric spaces.

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Geometric Analysis, Mathematical Relativity, and Nonlinear Partial Differential Equations

Preface

This volume presents the proceedings of the Southeast Geometry Seminar for the meetings that took place between the Fall of 2009 and the Fall of 2011. The one-day events were held bi-annually, and their location was rotated among four hosting institutions: Emory University, Georgia Institute of Technology, University of Alabama Birmingham, and the University of Tennessee.

The conference series has been running continuously since the spring of 2002. During this period, the meetings grew from a dozen participants to about 30 participants during the last meetings. The conference attracts researchers and students working in Geometric Analysis and related fields. The original rationale for the series was that since geometric analysis research groups in the region are typically small, short and frequent meetings which are accessible by road travel would foster collaboration and help these groups grow and attract students.

Topics for the conference series include geometric analysis and related fields, in particular, nonlinear partial differential equations, general relativity, and geometric topology. Geometric analysis has seen several major developments in the last two decades. Among these are a number of important breakthroughs, such as the global stability of Minkowski spacetime and Perelman's work on Hamilton's Ricci flow. These developments have generated a lot of activity, pushing the discipline forward. The field also has numerous applications outside geometry in many diverse fields, such as general relativity, quantum field theory, capillarity, topology, optics, microbiology, and image processing. The area holds the promise of many more exciting developments in the next few years.

The Southeast Geometry Seminar received financial support from the National Science Foundation (grant DMS-0940878). The organizational assistance provided by the hosting institutions, and in particular by the grant administrator, Ms. Cheryl Logan at the University of Alabama, was also essential for its success.

Fernando Schwartz
Knoxville, TN
March 2013

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On Dark Matter, Spiral Galaxies, and the Axioms of General Relativity

Hubert L. Bray

ABSTRACT. We define geometric axioms for the metric and the connection of a spacetime where the gravitational influence of the connection may be interpreted as dark matter. We show how these axioms lead to the Einstein-Klein-Gordon equations with a cosmological constant, where the scalar field of the Klein-Gordon equation represents the deviation of the connection from the standard Levi-Civita connection on the tangent bundle and is interpreted as dark matter.

This form of dark matter, while not quantum mechanical, gives virtually identical predictions to some other scalar field dark matter models, including boson stars, which others have shown to be compatible with the ΛCDM model on the cosmological scale. In addition, we quantify the already known fact that this scalar field dark matter, unlike the WIMP model, is automatically cold in a homogeneous, isotropic universe, sufficiently long after the Big Bang.

With these motivations in mind, we show how this scalar field dark matter, which naturally forms dark matter density waves due to its wave nature, may cause the observed barred spiral pattern density waves in many disk galaxies and triaxial shapes with plausible brightness profiles in many elliptical galaxies. If correct, this would provide a unified explanation for spirals and bars in spiral galaxies and for the brightness profiles of elliptical galaxies. We compare the results of preliminary computer simulations with photos of actual galaxies.

1. Overview

There are three ideas, each of independent interest, which, if correct, would create a new connection between differential geometry and astronomy, very much in the tradition of general relativity's previous successes at describing the large-scale structure of the universe. We begin by discussing these ideas in general terms.

Idea 1: Natural geometric axioms motivate studying the Einstein-Klein-Gordon equations with a cosmological constant. The Klein-Gordon equation is a wave type of equation for a scalar field which we propose as a model for dark matter. While this geometric motivation is new, modeling dark matter with a scalar field satisfying the Klein-Gordon equation is not (see [**18**] for a survey of scalar field dark matter and boson stars). Hence, ideas 2 and 3 apply to these other works as well.

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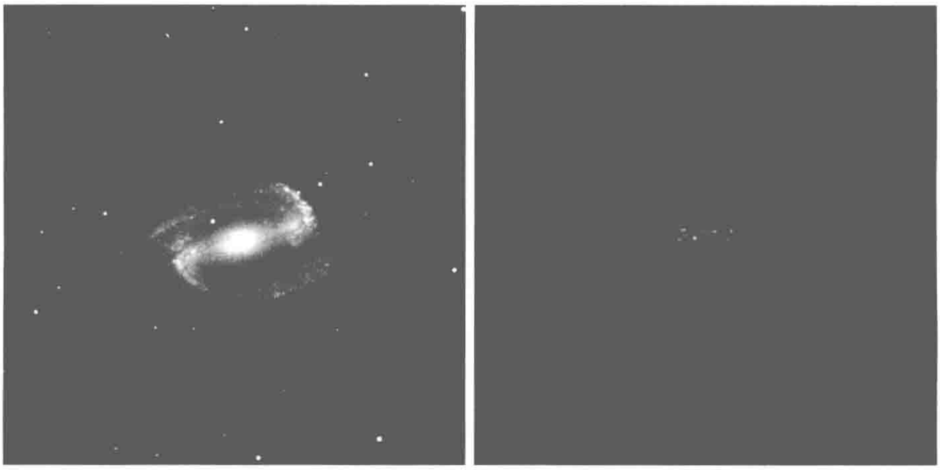

FIGURE 1. NGC1300 on the left, simulation on the right. The simulated image on the right results from running the Matlab function spiralgalaxy(1, 75000, 1, -1, 2000, 1990, 25000000, 8.7e-13, 7500, 5000, 25000000, 10000) described in section 5.1. Left photo credit: Hillary Mathis/NOAO/AURA/NSF. Date: December 24, 2000. Telescope: Kitt Peak National Observatory's 2.1-meter telescope. Image created from fifteen images taken in the BVR pass-bands.

Idea 2: Wave types of equations for matter fields, such as the Klein-Gordon equation, naturally form density waves in their matter densities because of constructive and destructive interference, like waves on a pond, or the Maxwell equations for electromagnetic radiation. Unlike waves on a pond or the Maxwell equations, however, the group velocities of wave solutions to the Klein-Gordon equation (with positive "mass" term) can be anything less than the speed of light, and can be arbitrarily slow for wavelengths which are long enough. This allows for the possibility of gravitationally bound "blobs" of dark matter to form. In this paper we make some conjectures about these scalar field dark matter density waves.

Idea 3: Density waves in dark matter, through gravity, naturally form density waves in the regular baryonic matter. In the case of disk galaxies, where friction in the interstellar medium of gas and dust is important [**9**], we exhibit examples where barred spiral density wave patterns form in the regular matter, as seen in figures 1, 2, 3, and 4. In the case of elliptical galaxies, where the interstellar medium is believed to be mostly irrelevant [**9**], we show how these dark matter density waves tend to produce triaxial ellipsoidal shapes for the regular matter with plausible brightness profiles, as seen in figures 5, 6, and 7.

This paper is an attempt to put the above three ideas together in as precise a way as possible. The rules of the game we choose to play here are strict: define a concise set of geometric axioms, and then try to understand the implications of those axioms. The axioms we choose, stated in the next section, are more fundamental than defining an action. Instead, we declare the spacetime metric and connection on the tangent bundle as the fundamental objects of our universe, and then define the properties that the action for these two objects must have. In this paper we

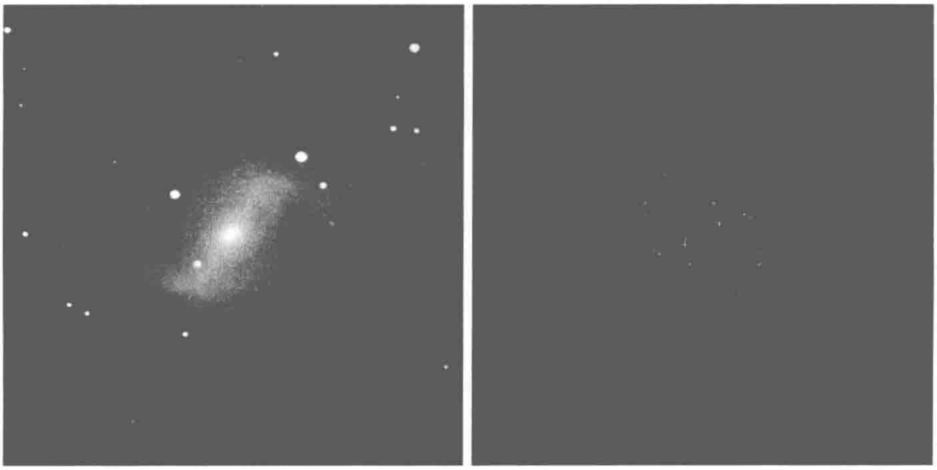

FIGURE 2. NGC4314 on the left, simulation on the right. The simulated image on the right results from running the Matlab function spiralgalaxy(1, 75000, 1, -0.15, 2000, 1990, 25000000, 8.7e-13, 7500, 5000, 30000000, 50000) described in section 5.4. Left photo credit: G. Fritz Benedict, Andrew Howell, Inger Jorgensen, David Chapell (University of Texas), Jeffery Kenney (Yale University), and Beverly J. Smith (CASA, University of Colorado), and NASA. Date: February 1996. Telescope: 30 inch telescope Prime Focus Camera, McDonald Observatory. `http://www.spacetelescope.org/`

make the case that the simulated images in figures 1-6 as well as the computed brightness profile described by figure 7 are all consequences of these axioms.

The resulting theory is a generalization of the vacuum Einstein equations with a cosmological constant, which already famously explains gravity, 73% of the mass of the universe as dark energy [**14**], the accelerating expansion of the universe, black holes, and all other vacuum general relativity effects. If successful, this generalized theory would then, by describing dark matter, account for 95% of the mass of the universe [**14**] and explain some portion of the structures of galaxies.

However, strictly speaking, quantum mechanics is not part of the theory we propose here, as should be expected since general relativity and quantum mechanics have yet to be unambiguously unified. Hence, our theory is clearly incomplete. This is not so bad considering that every theory known today is, in the strictest sense, incomplete. However, it is still a reasonable question to wonder if our theory does a good job of describing dark matter, even if it does not describe regular particulate matter. Hence, as a way of testing our dark matter model, we treat the remaining 4% of the mass of the universe, composed of particles of various kinds, in the traditional manner similar to test particles, but with mass, which arguably makes this theory compatible with ΛCDM models, as will be explained in section 3. The simulated images in figures 1-6 are pictures of the effect of the dark matter on the regular matter that we have sprinkled into the theory. Finding a less contrived way of getting regular matter into the next theory, while still respecting the idea of keeping our axioms as simple as possible, is an important open problem.

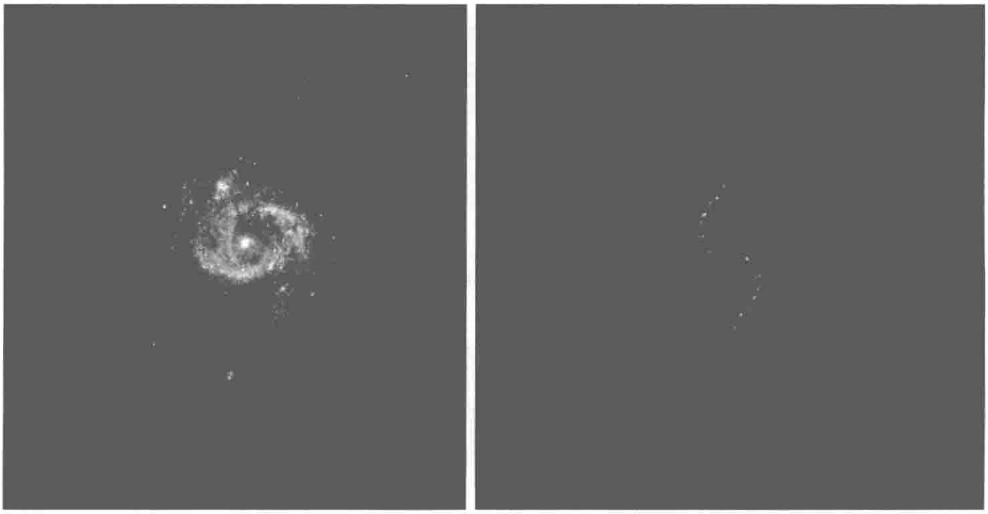

FIGURE 3. NGC3310 on the left, simulation on the right. The simulated image on the right results from running the Matlab function spiralgalaxy(1, 75000, 1, -0.15, 2000, 1990, 100000000, 8.7e-13, 7500, 5000, 45000000, 50000) described in section 5.5. Left photo credit: NASA and The Hubble Heritage Team (STScI/AURA). Acknowledgment: G.R. Meurer and T.M. Heckman (JHU), C. Leitherer, J. Harris and D. Calzetti (STScI), and M. Sirianni (JHU). Dates: March 1997 and September 2000. Telescope: Hubble Wide Field Planetary Camera 2.

So, does the Klein-Gordon equation accurately describe dark matter and predict some observed properties and structures of galaxies? In this paper we present evidence of this possibility by trying to understand the effect that this model of dark matter would have on the structure of galaxies, which is a reasonable idea since galaxies have large components of dark matter.

In doing so we have had to make approximations and educated guesses, so the comparisons in figures 1-6, while encouraging, should be taken in this context. Also, our "simulations" of galaxies only simulate the effect of the dark matter on the regular matter and hence are very primitive. Perhaps a better name would be "numerical experiments." However, one has to start somewhere, and it is already interesting that compelling patterns very much resembling actual galaxies have emerged. We will describe the models we have used in sections 4, 5, and 6 and the assumptions we have made. We will do our best to clearly label where we have had to make approximations and educated guesses, as well as rigorous arguments, so that readers may make their own judgments about what is presented here.

2. Geometric Motivation

Einstein's theory of general relativity was made possible by Gauss and Riemann who, decades before, began the field of mathematics now called differential geometry. Since then, advances in differential geometry have played a crucial role in understanding the implications of Einstein's theory. Einstein used differential geometry to make the qualitative statement "matter curves spacetime" precise,

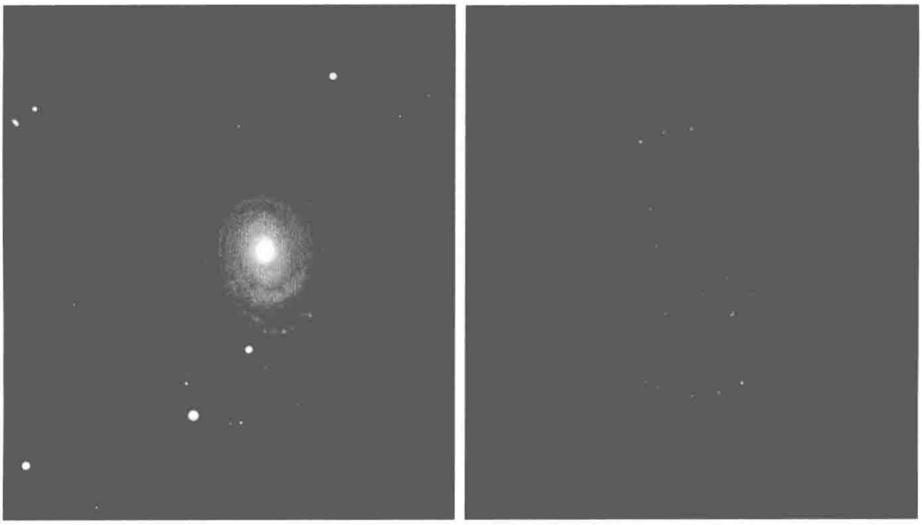

FIGURE 4. NGC488 on the left, simulation on the right. The simulated image on the right results from running the Matlab function spiralgalaxy(0.1, 100000, 1, -0.5, 20000, 19900, 50000000, 8.7e-15, 15000, 5000, 82000000, 20000) described in section 5.6. Left photo credit: Johan Knapen and Nik Szymanek. Telescope: Jacobus Kapteyn Telescope. B, I, and H-alpha bands.

thereby showing that gravity results as a consequence of this fundamental idea. In contrast, Newton's inverse square law for gravity, while a great approximation in the low-field limit, has been shown to be false by measuring the precession of the orbit of Mercury, for example, as well as the bending of light around the Sun, which is twice what is predicted by Newtonian physics and exactly what is predicted by general relativity. Hence, understanding gravity would appear to require differential geometry. In light of this rich history of differential geometry playing a vital role in understanding gravity and the large scale structure of the universe, it seems reasonable, among other ideas, to look for geometric motivations for dark matter.

The beginning point for our theory is to remove the assumption that the connection on the tangent bundle of the spacetime, an intrinsic geometric object second only to the metric in importance, is the standard Levi-Civita one. We compare this step to the jump from special relativity to general relativity, where the assumption that the metric of the spacetime is the standard flat one is removed. Our axioms then define the geometric properties that our action, which is now a function of the metric and the connection, must have.

We note that Einstein and Cartan famously played around with ideas similar to these by removing the assumption that the connection was torsion free, while still assuming metric compatibility. However, as our beginning point, we make neither assumption. Also, Einstein and Cartan were not trying to describe dark matter and thus had different objectives in mind.

Throughout this paper, the fundamental objects of our universe will be a spacetime manifold N with a metric g of signature $(-+++)$ and a connection ∇. We will assume that N is a smooth manifold which is both Hausdorff and second countable,

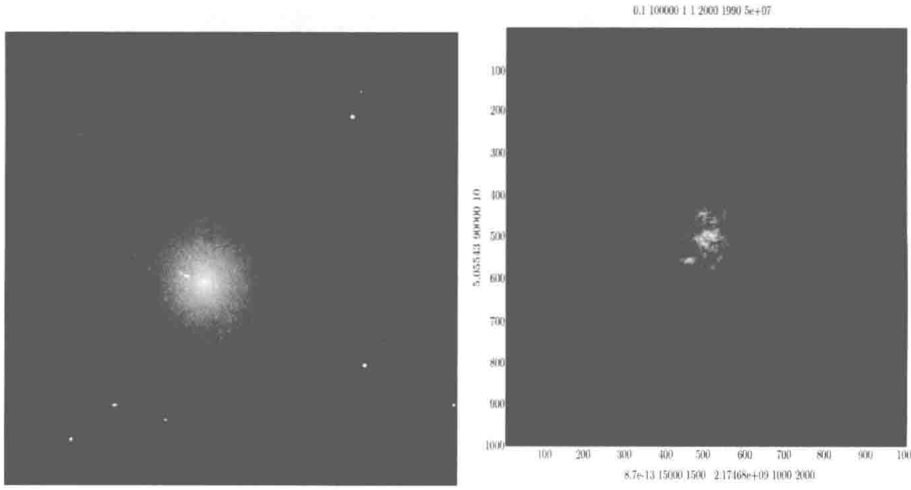

FIGURE 5. M87 on the left, simulation on the right. The simulated image on the right results from running the Matlab function ellipticalgalaxy(0.1, 100000, 1, 1, 2000, 1990, 50000000, 8.7e-13, 15000, 1500, 2174670000, 1000, 2000) described in section 6.1. Note that this simulated image is the top view of the same simulation in the next figure which shows the side view. Left photo credit: ING Archive and Nik Szymanek. Date: 1995. Telescope: Jacobus Kapteyn Telescope. Instrument: JAG CCD Camera. Detector: Tek. Filters B, V, and R.

which, while standard, deserves contemplation, as do all assumptions. We refer the interested reader to [**26**] as an excellent reference for the fundamentals of differential geometry. We will also assume that g and ∇ are smooth. These preliminary assumptions could be considered our "Axiom 0."

A smooth manifold N is a Hausdorff space with a complete atlas of smoothly overlapping coordinate charts [**26**]. Hence, we see that coordinate charts are more than convenient places to do calculations, but are in fact a necessary part of the definition of a smooth manifold. Given a fixed coordinate chart, let $\{\partial_i\}$, $0 \leq i \leq 3$, be the tangent vector fields to N corresponding to the standard basis vector fields of the coordinate chart. Let $g_{ij} = g(\partial_i, \partial_j)$ and $\Gamma_{ijk} = g(\nabla_{\partial_i}\partial_j, \partial_k)$, and let

$$M = \{g_{ij}\} \quad \text{and} \quad C = \{\Gamma_{ijk}\} \quad \text{and} \quad M' = \{g_{ij,k}\} \quad \text{and} \quad C' = \{\Gamma_{ijk,l}\}$$

be the components of the metric and the connection in the coordinate chart and all of the first derivatives of these components in the coordinate chart. We are now ready to state our central geometric axiom which motivates the remainder of this paper.

AXIOM 1. *For all coordinate charts $\Phi : \Omega \subset N \to R^4$ and open sets U whose closure is compact and in the interior of Ω, (g, ∇) is a critical point of the functional*

$$F_{\Phi,U}(g, \nabla) = \int_{\Phi(U)} Quad_M(M' \cup M \cup C' \cup C)\, dV_{R^4} \tag{1}$$

with respect to smooth variations of the metric and connection compactly supported in U, for some fixed quadratic functional $Quad_M$ with coefficients in M.