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新型有限元论

——结构工程中的高等有限元方法

Advanced Finite Element Method in Structural Engineering

清华大学出版社
Springer

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内 容 简 介

本书是在中文专著《新型有限元论》(2004 年版)的基础上补充了 2004 年至 2008 年期间的新成果所撰写的英文专著，是龙驭球院士、岑松博士和龙志飞教授及其研究组多年来在新型有限元方面研究成果的系统论述。全书分为 20 章。除首尾两章外，其余 18 章分为 3 篇：第 1 篇是变分原理进展，介绍分区和含参变分原理 2 项成果；它们为构造新型有限元起到理论指导作用。第 2 篇是有限元法进展初论，重点介绍广义协调元；这是在协调元与非协调元之间另辟的新路，使协调问题和收敛问题得到合理解决，单元构造方案可以灵活优选，学科内容得到充实更新；广义协调元是新型有限元方面的主要成果，在本书中起核心作用。第 3 篇是有限元法进展续论，补充介绍 4 项成果，包括分区混合元法、解析试函数法、第一和第二类四边形面积坐标法和样条函数有限元法，在本书中起锦上添花作用。本书还结合 7 项成果的论述，介绍了总共 108 个相关的新单元。

本书可作为高等学校力学、土木、机械等专业研究生和高年级本科生的教材和参考书，也可供相关领域教师和科技人员参考。

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Professor Yu-Qiu Long (1926—) is a Full Professor in the Department of Civil Engineering, Tsinghua University, and also a Member of the Chinese Academy of Engineering since 1995. During the past 60 years, he has been recognized as an expert in the areas of structural mechanics, shell structures, finite element method and variational principle. He was the first President of the Chinese Association of Structural Engineering between 1998 and 2003; Council Member of the Chinese Association of Computational Mechanics from 1957 to 1990; and Editor-in-Chief of a Chinese journal, *Engineering Mechanics*, between 1991 and 1999. He is currently a member of Editorial Boards of many international and Chinese journals, such as *Advances in Structural Engineering* (since 1997), *International Journal of Structural Stability and Dynamics* (since 2001), *Applied Mathematics and Mechanics* (since 1979), and so on. He published his first book on the finite element method in 1978, which is one of the earliest books on this subject in China and has a broad influence on Chinese readers.

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Professor Zhi-Fei Long (1957—) is a professor in the School of Mechanics & Civil Engineering, China University of Mining & Technology (Beijing). He is an expert in finite element method and structural mechanics, and has published 4 books and more than 90 papers in related fields. He won the Science Award of China Ministry of Education (First Class, 1993), for the studies on the generalized energy principles and new finite element models. The textbook titled by New Monograph of Finite Element Method (written by Zhi-Fei Long and Song Cen, published by China Hydraulic and Water-power Press in 2001) has produced a broad influence in Chinese universities.

Preface

The main purpose of this book is to describe some developments in finite element method and related variational principles. Since this book only deals with the areas the authors are familiar with, it is impossible to cover every aspect of these subjects. This book is composed of 20 chapters. Except for *Introduction* (Chap. 1) and *Concluding Remarks* (Chap. 20), in the other 18 chapters, seven theoretical achievements (two achievements in variational principles and five achievements in finite element methods are introduced, which are subdivided into three Parts.

Part Ⅰ focuses on advances in the variational principles. Two innovations in this subject are discussed here.

(1) *Sub-region variational principles* (Chap. 2). The concept of *sub-region* is introduced for establishing new variational principles suitable for the developments of the finite element method.

(2) *Variational principles with several adjustable parameters* (Chap. 3). Several adjustable parameters are included in the variational principles so that a broader optimization space is available.

Part Ⅱ focuses on the main advances in the finite element method—generalized conforming elements (the third innovation). Eight chapters are employed to illustrate this innovation.

(3) *Generalized conforming elements* (Chaps. 4–11). Firstly, from the viewpoint of theory, the generalized conforming element opens a new way between conforming and non-conforming elements, so that the puzzle of the convergence problem for non-conforming elements can be rationally solved. Meanwhile, various new conforming schemes, including point conforming, line conforming, perimeter conforming, SemiLoof conforming, least square conforming and their combination forms, have been successfully proposed. Secondly, from the viewpoint of applications, the successful application of the generalized conforming element method was first realized for thin plate bending problem, in which a series of high performance thin plate element models were presented. Subsequently, the novel technique was successfully generalized to other fields, and a large number of new models, including membrane elements, membrane elements with drilling DOFs, thin-thick plate elements, laminated composite plate elements, flat-shell

elements, curved shell elements, etc., were also successfully constructed.

Part III focuses on the other advances in the finite element method. Eight chapters are employed to discuss four additional subject innovations.

(4) *Sub-region mixed element method* (Chaps. 12 – 13). It provides a novel solution strategy for fracture problem by complementarity and coupling of displacement-based element and stress-based element.

(5) *Analytical trial function method* (Chaps. 14 – 15). This method exhibits rewarding cooperation between analytical and discrete methods, and provides effective solution strategy for shear locking, trapezoidal locking, and singular stress problems.

(6) *Quadrilateral area coordinate method* (Chaps. 16 – 17). This method indicates that the area coordinate method is generalized from the traditional triangular element field to new fields.

(7) *Spline element method* (Chaps. 18 – 19). This method indicates that the advantages of the spline functions have been adopted by the finite element method.

While introducing above seven theoretical innovations, five new element series with 108 new element models, which were directly derived from the five achievements in FEM, are also discussed in detail or briefly (see Table 20.2). Furthermore, based on these developments, effective solution strategies for five challenging problems (shear-locking problem in thick plate elements, sensitivity problem to mesh distortion, non-convergence problem of non-conforming elements, accuracy loss problem of stress solutions by displacement-based elements, and singular stress problem) have also been found.

To sum up, in the contents of this book, three aspects should be emphasized:

(1) Seven new achievements in the field of variational principle and FEM;

(2) five new element series with 108 new element models;

(3) five sets of novel solution strategies for five challenging problems.

The authors are very grateful to all the colleagues and students who made significant contributions to the contents included in this book. We also thank China Academy of Building Research for compiling our algorithms and finite element models into their FEM software product, SATWE, for designs of high-rise building structures.

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Chapter 1 Introduction—The Evolutive Finite Element Method

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Abstract This chapter is an opening introduction to the entire book, and also an introduction to the evolutive Finite Element Method (FEM). Firstly, a brief review on the features of FEM is given. Then, a close relationship between FEM and variational principle is discussed according to the development history and categories of FEM. Thirdly, some research areas of FEM of significant interest are listed. Finally, the topics of the book are presented. The purpose of the above arrangement is to explain the background and main idea of this book.

Keywords finite element method, variational principle, research area, advance, outline.

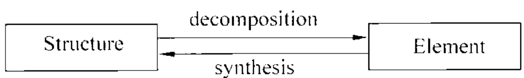
1.1 Brief Review of the Features of Finite Element Method

Computational mechanics is a flourishing subject for science and engineering, in which the physical mechanics problems are solved by cooperation of mechanics, computers and various numerical methods. It has already entered every branch of mechanics, and is being generalized continuously for broader research and application ranges. At the same time, new theories and methods of computational mechanics itself are also being developed gradually.

Finite element method is an important branch of computational mechanics. It is a kind of numerical methods in which various mechanics problems are solved by discretizing related continuums. In 1960, R. W. Clough firstly used the name of *Finite Element Method*. Up to date, it has already been one of the most powerful techniques for dealing with problems in mechanics, physics and engineering computations.

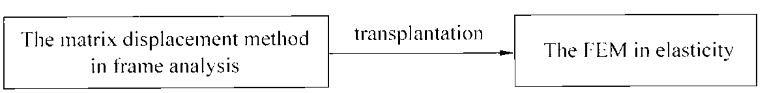
1.1.1 Features of FEM from the Viewpoints of Methodology

(1) FEM is an application of both methods of analysis and synthesis: during the procedure, one structure will be firstly decomposed into elements, and then, these elements are synthesized to be the structure again. Solutions for the structure problem can be obtained from such decomposition and synthesis.

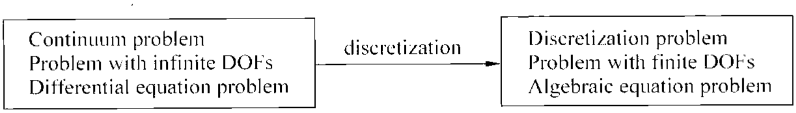


Decomposition — breaks up the whole into parts, so that difficulties can be transformed into simplicity
Synthesis — integrates all parts into a whole, so that the prototype can be recovered

(2) From the viewpoint of evolution of mechanics, it can be seen that FEM evolves from the matrix displacement method for frame analysis. Along with the transplantation from frame analysis to elasticity, the matrix displacement method becomes FEM.



(3) From the viewpoint of mathematics, it can be seen that FEM is a discrete approximation for continuum problems. Thus, the original problems with infinite degrees of freedom (DOFs) can be approximately treated as those with finite DOFs. Corresponding differential equations can also be simplified into algebraic ones.



1.1.2 Features of FEM from the Viewpoint of Solution Method Classification

The solution methods for problems in mechanics can be classified into following three categories:

(1) Analytical methods;

(2) Numerical methods;

(3) Semi-analytical methods.

The usual numerical methods are as follows (FEM is one of them—it is not the oldest one but exerts the greatest influence):

(1) Finite difference method—the differential equations are transformed into difference forms so that approximate solutions can be obtained.

(2) Weighted residual method—the differential equations are transformed into weighted integration forms so that approximate solutions can be obtained. In such method, there are five usual schemes: collocation method, sub-domain method, least square method, Galerkin method, and method of moment.

(3) Finite element method—problems related to the differential equations are transformed into those related to stationary values of energy, and sub-region interpolation technique [interpolation is performed in each sub-region (element)] is used, so that approximate solutions can be obtained.

(4) Boundary element method—discretization is performed only at boundaries.

(5) Mesh-free method—its approximate functions are mainly established at discrete points; thus, no mesh is needed.

The usual semi-analytical methods are as follows:

(1) Kantorovich method;

(2) Finite strip method;

(3) Finite element method of line.

1.2 Finite Element Method and Variational Principles

Finite element method has a close relationship with variational principles. Here, this relationship is discussed according to its development history and categories.

1.2.1 Creation of FEM and Variational Principles

It is well recognized in academia that the variational energy principle is the basis of FEM. However, the path to this recognition was not smooth.

Firstly, in the field of applied mathematics, the first paper published on FEM was the report, *Variational methods for the solution of problems of equilibrium and vibration*[1], delivered by Courant in 1941 and published in 1943. He used the variational principle and sub-region interpolation technique to look for the approximate solutions of torsion problem. In the title of his paper, he named his method, which is called *finite element method* afterwards, as *variational method*. Since computers have not been available at that time, this paper did not attracted due attention.

Secondly, in the field of engineering techniques, Turner, Clough, Martin and Toop published the first paper about FEM, entitled *stiffness and deflection analysis of complex structures*[2], in 1956. They generalized the matrix displacement method of rigid frame to the plane problem in elasticity. In the title of the paper, they named such solution scheme as the *stiffness method for complex structures* (or *direct stiffness method*), and subsequently, clough denominated it as Finite Element Method[3]. Like all engineers at that time, these authors did not pay much attention to Courant's paper which had been ignored, and did not pay attention either to the relation between the direct stiffness method and Courant's variational method.

Finally, some related papers began to appear in 1963, including the paper, *Basis for derivation for the Direct Stiffness Method*, by Melosh[4]. These papers drew an important conclusion that, the basis of the direct stiffness method (i.e. finite element method) is just the variational principle, or, the direct stiffness method is a new Ritz method which is based on the variational principle (new Ritz method using sub-region interpolation technique). In this way, a bridge was successfully built that linked mathematics and engineering. Consequently, FEM is recognized as a numerical method with rigorous theoretical basis and universal application value. Systematical presentation of FEM is given in books and monographs [5–14].

1.2.2 Element Categories and Variational Principles

Different elements are derived from different variational principles. Several element categories and their corresponding variational principles are as follows.

(1) Conforming displacement-based element (its displacement trial functions exactly conform between two adjacent elements)—the minimum potential energy principle.

(2) Non-conforming displacement-based element (its displacement trial functions do not conform exactly between two adjacent elements)—the sub-region potential energy principle.

(3) Generalized conforming displacement-based element (its displacement trial functions are generalized conforming between two adjacent elements)—the degenerated form of sub-region potential energy principle.

(4) Hybrid stress-based element (the stress trial functions satisfying the equilibrium differential equation are used)—the minimum complementary energy principle.

(5) Mixed element (mixed trial functions containing displacements, stresses and strains are used)—generalized variational principle.

(6) Sub-region mixed element (some elements utilize the displacement trial functions, and the other use the stress trial functions)—the sub-region mixed energy principle.

1.3 Research Areas of FEM

During the research history of theories and applications of FEM, the following problems have attracted main attention.

(1) Variational principle and numerical method

In the development of variational principle and FEM, two classical papers should be mentioned: the generalized variational principle proposed by Hu in 1955[15] which provided a theoretical basis for development of FEM; and the difference formulations based on variational principle proposed by Feng[16], which is virtually the modern FEM.

New forms of variational principles were proposed in order to satisfy the requirements caused by the development of FEM, such as sub-region potential, complementary and mixed energy principles, the degenerated forms of sub-region variational principles and their applications, variational principles with adjustable parameters and their applications, variational principles for piezoelectric composite structures, variational principles for micromechanics based on the strain gradient theory, and so on. The attention was frequently focused on error estimation, convergence, reliability, self-adaptation and optimization of related numerical methods.

(2) Construction techniques for new elements

The existing construction modes, such as hybrid element, mixed element, quasi-conforming element, strain-based element, spline element, and so on, were extended.

The new construction modes, such as generalized conforming element based on generalized conforming theory, sub-region mixed element based on sub-region mixed variational principle, rational element, new quadrilateral element based on quadrilateral area coordinates, element based on analytical trial functions, were successfully developed.

(3) Challenging problems and their solution strategies

The development of FEM still left some difficult and challenging problems. Some of them have remained unsolved for a long time. Naturally, these unsolved challenging problems became the focus of attention, such as various locking phenomena (shear locking, membrane locking, bulk locking), sensitive problem to mesh distortion, non-convergence problem of some non-conforming elements, spurious zero energy mode, solution oscillation phenomena, accuracy loss problem (stress solutions of displacement-based elements, transverse shear stresses of laminated composite plate), singular stress problem, ill-conditioned phenomena in numerical computations, and so on.

(4) Complicated problems

Complicated problems include finite element analysis of problems with material and geometric nonlinearity, buckling analysis of shell structures, finite element analysis of plastic forming, numerical simulation of impact problem, finite element method based on the strain gradient theory, and so on.