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shôichirô Sakai

C*-Algebras and W*-Algebras

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Shôichirô Sakai

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Preface

The theory of operator algebras in a Hilbert space was initiated by von Neumann [126] in 1929. In the introduction to the paper [119] in 1936, Murray and von Neumann stated that the theory seems to be important for the formal calculus with operator-rings, the unitary representation theory of groups, a quantum mechanical formalism and abstract ring theory. These predictions have been completely verified. Furthermore the theory of operator algebras is now becoming a common tool in a number of fields of mathematics and theoretical physics beyond those mentioned by Murray and von Neumann. This is perhaps to be expected, since an operator algebra is an especially well-behaved infinite-dimensional generalization of a matrix algebra. Therefore one can confidently predict that the active involvement of operator algebra theory in various fields of mathematics and theoretical physics will continue for a long time.

Another application of the theory is to the study of a single operator in a Hilbert space (see, for example, [99]). Nowadays we may add further the theory of singular integral operators and K-theory as other applications. Such diversifications of the theory of operator algebras have already made a unified text book concerning the theory virtually impossible. Therefore I have no intention of giving a complete coverage of the subject. I will rather take a somewhat personal stand on the selection of material—i.e., the selection is concentrated heavily on the topics with which I have been more or less concerned (needless to say, there are many other very important contributors to those topics. The reader will find the names of authors who have made remarkable contributions to those discussed in the concluding remarks of each section). Consequently parts of the book tend to be somewhat monographic in character.

Let me explain briefly about the contents. There are essentially two different ways of studying the operator *-algebras in Hilbert spaces. The first alternative is to assume that the algebra is weakly closed (called a W^*-algebra). These algebras are also called Rings of operators, and more recently, von Neumann algebras.

The earliest study along this line is due to von Neumann in 1929. In a series of five memoirs beginning with [119], Murray and von Neumann laid the foundation for the theory of W^*-algebras. Virtually all of the later work on these algebras is based directly or indirectly on their pioneering work.

We call a W^*-algebra a factor if its center is just the complex numbers. Murray and von Neumann concentrated most of their attention on factors. However, von Neumann [132] obtained a reduction theory by which the study of a general W^*-algebra may be to a large extent reduced to the case of a factor. At the same time a number of authors have pushed through the major portions of a global theory for general W^*-algebras (the reader may find a long list of papers by many authors in the bibliography in [37]).

The second alternative is to assume only that the algebra is uniformly closed (called a C^*-algebra). The earliest study along this line is due to Gelfand and Naimark [55] in 1943. A notable advantage of the C^*-algebra is the existence of an elegant system of intrinsic postulates, formulated by Gelfand and Naimark, which gives an abstract characterization of these algebras.

Using this approach, Segal [180] in 1947 initiated a study of C^*-algebras. The subsequent development which contains many beautiful results (cf. Chapters 1, 3, & 4) has been carried out by a number of authors.

The theory of C^*-algebras fits naturally into the theory of Banach algebras, and in certain respects they are among the best behaved examples of infinite dimensional Banach algebras.

In chapter 1, the characterization of W^*-algebras obtained in [149] is used to define W^*-algebras as abstract Banach algebras, like C^*-algebras, and to develop the abstract treatments of both W^*- and C^*-algebras.

Chapter 2 is concerned mainly with the classification and representation theory of W^*-algebras which are developed along classical standard lines. Some proofs may be new.

In chapter 3, the reduction theory is discussed. Here, a modern method, developed recently in [146], [159], [218] (i.e. the decomposition theory of states) is used. Also discussed are some recent results obtained by theoretical physicists.

Chapter 4 consists of some special topics from the theories of W^*-algebras and C^*-algebras. This chapter is the most personal in the book. All the topics covered are ones with which I have been more or less concerned. They are: Derivations and automorphisms on operator algebras; examples of factors; examples of non-trivial global W^*-algebras; type I C^*-algebras and a Stone-Weierstrass theorem for C^*-algebras.

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Philadelphia, Pa. U.S.A., January 1971 S. Sakai

Contents

1. General Theory

1.1. Definitions of C^*-Algebras and W^*-Algebras

Let $\mathscr{A}$ be a linear associative algebra over the complex numbers. The algebra $\mathscr{A}$ is called a normed algebra if there is associated to each element x a real number $\|x\|$, called the norm of x, with the properties:

I $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;
II $\|x+y\| \leq \|x\| + \|y\|$;
III $\|\lambda x\| = |\lambda| \, \|x\|$, λ a complex number;
IV $\|xy\| \leq \|x\| \, \|y\|$.

If $\mathscr{A}$ is complete with respect to the norm (i.e. if $\mathscr{A}$ is also a Banach space), then it is called a Banach algebra. A mapping $x \to x^*$ of $\mathscr{A}$ into itself is called an involution if the following conditions are satisfied

I $(x^*)^* = x$;
II $(x+y)^* = x^* + y^*$;
III $(xy)^* = y^* x^*$;
IV $(\lambda x)^* = \bar{\lambda} x^*$, λ a complex number.

An algebra with an involution $*$ is called a $*$-algebra.

1.1.1. Definition. *A Banach $*$-algebra $\mathscr{A}$ is called a C^*-algebra if it satisfies $\|x^* x\| = \|x\|^2$ for $x \in \mathscr{A}$.*

1.1.2. Definition. *A C^*-algebra $\mathscr{M}$ is called a W^*-algebra if it is a dual space as a Banach space (i.e., if there exists a Banach space $\mathscr{M}_*$ such that $(\mathscr{M}_*)^* = \mathscr{M}$, where $(\mathscr{M}_*)^*$ is the dual Banach space of $\mathscr{M}_*$). We shall call such a Banach space $\mathscr{M}_*$ the predual of $\mathscr{M}$.*

Remark. It is not true in general that a dual Banach space is the dual space of a unique Banach space. For example, c^* and c_0^* are isometrically isomorphic to l^1, but c is not isometrically isomorphic to c_0, where c is the Banach space of all convergent sequences, c_0 the Banach space of all sequences convergent to zero, and l^1 is the Banach space of all summable sequences. However, we shall show later that a W^*-algebra is the dual space of a unique Banach space.

1.1.3. Definition. *The topology defined by the norm* $\| \ \|$ *on a* C^**-algebra* $\mathscr{A}$ *is called the uniform topology. The weak* $*$*-topology* $\sigma(\mathscr{M}, \mathscr{M}_*)$ *on a* W^**-algebra* $\mathscr{M}$ *is called the weak topology or the* σ*-topology on* $\mathscr{M}$.

1.1.4. Definition. *A subset* V *of a* C^**-algebra* $\mathscr{A}$ *is called self-adjoint if* $x \in V$ *implies* $x^* \in V$. *A self-adjoint, uniformly closed subalgebra of* $\mathscr{A}$ *is also a* C^**-algebra. It is called a* C^**-subalgebra of* $\mathscr{A}$. *A self-adjoint,* σ*-closed subalgebra* $\mathscr{N}$ *of a* W^**-algebra* $\mathscr{M}$ *is also a* W^**-algebra, because* $(\mathscr{M}_*/\mathscr{N}^0)^* = \mathscr{N}$, *where* $\mathscr{N}^0$ *is the polar of* $\mathscr{N}$ *in* $\mathscr{M}_*$. $\mathscr{N}$ *is called a* W^**-subalgebra of* $\mathscr{M}$.

1.1.5. Definition. *Let* $\{\mathscr{A}_\alpha\}_{\alpha \in \mathbb{I}}$ *be a family of* C^**-algebras. The direct sum* $\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{A}_\alpha$ *of* $\{\mathscr{A}_\alpha\}_{\alpha \in \mathbb{I}}$ *is defined as follows: elements of* $\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{A}_\alpha$ *are composed of all families* $(a_\alpha)_{\alpha \in \mathbb{I}}$ *such that* $a_\alpha \in \mathscr{A}_\alpha$ *and* $\sup_\alpha \|a_\alpha\| < +\infty$, *and the operations:* $\lambda(a_\alpha) = (\lambda a_\alpha)$ *(*λ *a complex number),* $(a_\alpha) + (b_\alpha) = (a_\alpha + b_\alpha)$, $(a_\alpha)(b_\alpha) = (a_\alpha b_\alpha)$, $(a_\alpha)^* = (a_\alpha^*)$ *and* $\|(a_\alpha)\| = \sup_\alpha \|a_\alpha\|$. $\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{A}_\alpha$ *is again a* C^**-algebra.*

Let $\{\mathscr{M}_\alpha\}_{\alpha \in \mathbb{I}}$ be a family of W^*-algebras. The direct sum $\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{M}_\alpha$ is a dual space since $\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{M}_\alpha = \left(\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{M}_{\alpha *}\right)^*_{l_1}$, *where the norm of an* element $(f_\alpha)(f_\alpha \in \mathscr{M}_{\alpha *})$ in $\left(\sum_{\alpha \in \mathbb{I}} \oplus \mathscr{M}_{\alpha *}\right)_{l^1}$ is defined by $\|(f_\alpha)\| = \sum_{\alpha \in \mathbb{I}} \|f_\alpha\|$; hence it is a W^*-algebra.

1.1.6. Lemma. *Let* $\mathscr{A}$ *be a* C^**-algebra; then* $\|x\| = \|x^*\|$ *for* $x \in \mathscr{A}$.

Proof. $\|x\|^2 = \|x^* x\| \leq \|x^*\| \ \|x\|$; hence $\|x\| \leq \|x^*\|$ and analogously $\|x^*\| \leq \|x^{**}\| = \|x\|$. q.e.d.

1.1.7. Proposition. *Let* $\mathscr{A}$ *be a* C^**-algebra without identity, and let* $\mathscr{A}_1$ *be the algebra obtained from* $\mathscr{A}$ *by adjoining the identity* 1. *For* $x \in \mathscr{A}$, λ *a complex number, define* $\|\lambda 1 + x\| = \sup_{\|y\| \neq 0} \dfrac{\|\lambda y + xy\|}{\|y\|}$. *Then* $\mathscr{A}_1$ *is a* C^**-algebra with norm* $\| \ \|$.

Proof. It is easy to see that $\mathscr{A}_1$ is a Banach $*$-algebra. Suppose μ is an arbitrary positive number less than 1. Then there exists a $y \in \mathscr{A}$ such that $\|y\| = 1$ and $\mu \|\lambda 1 + x\| < \|\lambda y + xy\|$.

Then,

$$\begin{aligned} \mu^2 \|\lambda 1 + x\|^2 < \|\lambda y + xy\|^2 &= \|(\lambda y + xy)^* (\lambda y + xy)\| \\ &= \|y^* (\lambda 1 + x)^* (\lambda 1 + x) y\| \leq \|(\lambda 1 + x)^* (\lambda 1 + x)\| . \end{aligned}$$

Hence $\mu^2\|\lambda 1+x\|^2 \le \|(\lambda 1+x)^*(\lambda 1+x)\|$ and so

$$\|\lambda 1+x\|^2 \le \|(\lambda 1+x)^*(\lambda 1+x)\|;$$

this implies $\|\lambda 1+x\|^2 \le \|(\lambda 1+x)^*\|\,\|\lambda 1+x\|$ and so

$$\|\lambda 1+x\| \le \|(\lambda 1+x)^*\|;$$

therefore $\|(\lambda 1+x)^*\| \le \|(\lambda 1+x)^{**}\| = \|\lambda 1+x\|$; finally

$$\|\lambda 1+x\|^2 = \|(\lambda 1+x)^*(\lambda 1+x)\|. \qquad \text{q.e.d.}$$

Let $\mathscr{A}$ be a C^*-algebra with identity 1. The spectrum $\mathrm{Sp}(a)$ of an element a in $\mathscr{A}$ is the set of all complex numbers λ such that $x-\lambda 1$ is not invertible.

If $\mathscr{A}$ is a C^*-algebra without identity, the spectrum $\mathrm{Sp}(a)$ of an element a in $\mathscr{A}$ is the spectrum of a as an element of the C^*-algebra $\mathscr{A}_1$ obtained from $\mathscr{A}$ by adjoining the identity 1. In general, $\mathrm{Sp}(a)$ is a closed subset in the complex field.

An element a in $\mathscr{A}$ is called normal (resp. self-adjoint, unitary), if $a^*a=aa^*$ (resp. $a^*=a$, $a^*a=aa^*=1$).

A self-adjoint element e in $\mathscr{A}$ is called a projection if $e^2=e$. An element v is called a partial isometry if v^*v is a projection.

1.1.8. Proposition. $\mathrm{Sp}(ab)\cup(0)=\mathrm{Sp}(ba)\cup(0)$ *and* $\mathrm{Sp}(a^*)=\overline{\mathrm{Sp}(a)}$.

Proof. It is clear that $\mathrm{Sp}(a^*)=\overline{\mathrm{Sp}(a)}$. Suppose that $\lambda\neq 0$ and $ab-\lambda 1$ has the inverse u; then

$$\begin{aligned}(ba-\lambda 1)(bua-1) &= babua-ba-\lambda bua+\lambda 1\\ &= b(ab-\lambda 1)ua+\lambda bua-ba-\lambda bua+\lambda 1\\ &= ba-ba+\lambda 1=\lambda 1.\end{aligned}$$

Analogously, $(bua-1)(ba-\lambda 1)=\lambda 1$. Hence, $(ba-\lambda 1)$ is invertible and so $\mathrm{Sp}(ab)\cup(0)\supset\mathrm{Sp}(ba)\cup(0)$; proof of the reverse inclusion is analogous. q.e.d.

1.1.9. Corollary. *Let v be a partial isometry. Then vv^* is also a projection. v^*v is called the initial projection of v and vv^* is called the final projection* of v.

Remark. A C^*-algebra $\mathscr{A}$ can be defined by the following weaker condition: $\|x^*x\|=\|x^*\|\,\|x\|$ *for* $x\in\mathscr{A}$ (cf. [61], [136]).

References. [8], [17], [26], [28], [55], [124], [141], [149].

1.2. Commutative C^*-Algebras

Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras, and let Φ be a homomorphism of $\mathscr{A}$ into $\mathscr{B}$ (i.e., $\Phi(\lambda x)=\lambda\Phi(x)$, $\Phi(x+y)=\Phi(x)+\Phi(y)$, $\Phi(xy)=\Phi(x)\Phi(y)$ for a complex number λ and x, $y \in \mathscr{A}$). Φ is called a $*$-homomorphism if $\Phi(x^*) = \Phi(x)^*$; Φ is called an isomorphism (resp. a $*$-isomorphism) if it is a one-to-one homomorphism (resp. $*$-homomorphism). We say that a C^*-algebra $\mathscr{A}$ is $*$-isomorphic to a C^*-algebra $\mathscr{B}$ if there exists a $*$-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$.

Let K be a compact Hausdorff space, and let $C(K)$ be the algebra, under pointwise multiplication, of all complex valued, continuous functions on K. Define $\|a\| = \sup_{t \in K} |a(t)|$, and $a^*(t) = \overline{a(t)}$ for $a \in C(K)$. Then $C(K)$ is a commutative C^*-algebra.

1.2.1. Theorem. *Let $\mathscr{A}$ be a commutative C^*-algebra with identity. Then $\mathscr{A}$ is $*$-isomorphic to the C^*-algebra $C(K)$ of all complex valued, continuous functions on K, where K is the compact Hausdorff space of all maximal ideals of $\mathscr{A}$ (called the spectrum space of $\mathscr{A}$); therefore we can identify $\mathscr{A}$ with $C(K)$.*

Proof. By the commutative normed algebra theory of Gelfand (cf. [54]), there exists a homomorphism Φ of $\mathscr{A}$ into $C(K)$.

$$\begin{aligned}\|x^{2^n}\|^2 &= \|(x^{2^n})^*(x^{2^n})\| = \|(x^*x)^{2^n}\| \\ &= \|(x^*x)^{2^{n-1}}\|^2 = \cdots \\ &= \|x^*x\|^{2^n} = \|x\|^{2^{n+1}}.\end{aligned}$$

Hence $\|x\| = \lim_n \sqrt[2^n]{\|x^{2^n}\|} = \sup_{\lambda \in \mathrm{Sp}(x)} |\lambda| = \|\Phi(x)\|$. Therefore, Φ is isometric; hence it is one-to-one. Let h be a self-adjoint element in $\mathscr{A}$. Then $(\exp ih)^* = \exp(-ih)$ and so $(\exp ih)(\exp ih)^* = 1$. Hence

$$\|(\exp ih)^*\| \, \|\exp ih\| = 1;$$

also $\mathrm{Sp}((\exp ih)^*) = \overline{\mathrm{Sp}(\exp ih)}$; therefore $\lambda \in \mathrm{Sp}(\exp ih)$ implies $|\lambda| = 1$. Hence $\mathrm{Sp}(h)$ is a subset of the real numbers. Let $a = h + ik$ (h, k self-adjoint); then $\Phi(a^*) = \Phi(h - ik) = \Phi(h) - i\Phi(k)$ and so $\overline{\Phi(a)} = \Phi(a^*)$. q.e.d.

As a corollary.

1.2.2. Corollary. *Let $\mathscr{A}$ be a commutative C^*-algebra without identity. Then $\mathscr{A}$ is $*$-isomorphic to the C^*-algebra $C_0(\Omega)$ of all complex valued continuous functions which vanish at infinity on a locally compact Hausdorff space Ω (Ω is called the spectrum space of $\mathscr{A}$).*

1.2.3. Corollary. *Let $\mathscr{A}$ be a commutative C^*-algebra generated by a single self-adjoint element a, and let Ω be the spectrum space of $\mathscr{A}$. Then $\Omega \cup (\infty)$ is homeomorphic to $\mathrm{Sp}(a) \cup (0)$. Moreover, $\mathscr{A} = C_0(\mathrm{Sp}(a) \cup (0))$, where $C_0(\mathrm{Sp}(a) \cup (0))$ is the C^*-algebra of all complex valued continuous functions which vanish at 0 on the compact space $\mathrm{Sp}(a) \cup (0)$.*

Proof. Let Ω be the spectrum space of $\mathscr{A}$, and let $t \in \Omega$. Consider the mapping ξ of $\Omega \cup (\infty)$ onto $\mathrm{Sp}(a) \cup (0)$ defined by $\xi(t) = a(t)$ and $\xi(\infty) = 0$. ξ is one-to-one, since $\mathscr{A}$ is generated by a. ξ is continuous, and so it is a homeomorphism. q.e.d.

1.2.4. Theorem. *Let $\mathscr{A}$ be a commutative C^*-algebra, $\|\ \|_1$ another norm on $\mathscr{A}$ under which $\mathscr{A}$ is a normed algebra. Then $\|a\| \leq \|a\|_1$ for $a \in \mathscr{A}$.*

Proof. It suffices to prove the assertion in case $\mathscr{A}$ has an identity. Let $\mathscr{A} = C(K)$. Each point $t \in K$ will give a character of $\mathscr{A}$. Let K_1 be the set of all $\|\ \|_1$-continuous characters on $\mathscr{A}$; then $K_1 \subset K$. Let $\overline{K_1}$ be the closure of K_1 in K. Suppose that $\overline{K_1} \subsetneqq K$ and take an open set G such that $\overline{G} \subset \overline{K}_1^c$, where $\overline{K}_1^c$ is the complement of $\overline{K_1}$. Then there exists a continuous function b on K such that $b(t) = 1$ on $\overline{K_1}$ and $b(t) = 0$ on $\overline{G}$. Let $\tilde{\mathscr{A}}$ be the commutative Banach algebra obtained by the $\|\ \|_1$-completion of $\mathscr{A}$.

Suppose b is not invertible in $\tilde{\mathscr{A}}$; then there exists a $\|\ \|_1$-continuous character x_0 of $\tilde{\mathscr{A}}$ such that $b(x_0) = 0$. The restriction of x_0 on $\mathscr{A}$ will define a character t_0 belonging to K_1. Hence, $b(t_0) = 0$, a contradiction. Hence b is invertible in $\tilde{\mathscr{A}}$.

On the other hand, there exists an element c in $\mathscr{A}$ such that $c \neq 0$ and the support of $c \subset \overline{G}$; $bc = 0$ and so $c = 0$. Hence $\overline{K_1} = K$. Therefore $\|a\|_1 \geq \lim_n \sqrt[n]{\|a^n\|_1} = \sup_{t \in K_1} |a(t)| = \|a\|$. q.e.d.

1.2.5. Corollary. *Along with the assumptions of* 1.2.4, *suppose that $\|a^* a\|_1 = \|a\|_1^2$. Then $\|a\| = \|a\|_1$ for all $a \in \mathscr{A}$.*

Proof. $\|a\|_1 = \lim_n \sqrt[2^n]{\|a^{2^n}\|_1} = \sup_{t \in K_1} |a(t)| = \|a\|$. q.e.d.

1.2.6. Corollary. *Let $\mathscr{A}$, $\mathscr{B}$ be C^*-algebras and let Φ be a *-homomorphism of $\mathscr{A}$ into $\mathscr{B}$. Then $\|\Phi(x)\| \leq \|x\|$ for $x \in \mathscr{A}$. If Φ is a *-isomorphism, Φ is an isometry.*

Proof. Consider the direct sum $\mathscr{A} \oplus \mathscr{B}$ and the *-isomorphism Ψ of $\mathscr{A}$ into $\mathscr{A} \oplus \mathscr{B}$ defined as follows; $\Psi(a) = (a, \Phi(a))$ for $a \in \mathscr{A}$. Then

$$\|\Psi(a)\|^2 = \|\Psi(a)^* \Psi(a)\| = \|\Psi(a^* a)\| \leq \|a^* a\|$$

(by 1.2.5, since $\|b\|_1 = \max(\|b\|, \|\Psi(b)\|)$ for $b \in \mathscr{A}$ satisfies

$$\|b^* b\|_1 = \|b\|_1^2). \quad \text{q.e.d.}$$

Concluding remarks on 1.2

Theorem 1.2.1 is due to Gelfand [54]; Theorem 1.2.4 is due to Kaplansky [92].

References. [54], [92].

1.3. Stonean Spaces

Let K be a compact Hausdorff space. K is called a Stonean space if the closure of every open set is open.

1.3.1. Proposition. *Let K be a Stonean space. Then every element a in $C(K)$ can be uniformly approximated by finite linear combinations of projections.*

Proof. Let $a \in C(K)$. It suffices to assume that a is a positive function on K. For $\varepsilon > 0$, consider the finite division Δ of the interval

$$[0, \|a\| + 1]: 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \|a\| + 1$$

and $\lambda_{i+1} - \lambda_i < \varepsilon$ $(i = 1, 2, \ldots, n-1)$. By induction, we shall define a finite family $\{G_i\}_{i=1,2,\ldots,n}$ of open and closed sets.

$$G_1 = \text{the closure of } \{t \mid a(t) < \lambda_1, t \in K\},$$

$$G_i = \text{the closure of} \left\{t \mid a(t) < \lambda_i, t \in K - \bigcup_{j=1}^{i-1} G_j\right\}$$

$(i = 2, 3, \ldots, n)$.

Then $G_i \cap G_j = (\emptyset)$ if $i \neq j$. Let x_j be the characteristic function of G_j; then it is continuous and a projection.

Moreover

$$\left| a(t) - \sum_{i=1}^{n} \lambda_i x_i(t) \right| < \varepsilon \quad \text{for } t \in K;$$

hence $\left\| a - \sum_{i=1}^{n} \lambda_i x_i \right\| < \varepsilon$. q.e.d.

1.3.2. Proposition. *Let K be a compact Hausdorff space. Suppose every bounded increasing directed set of real valued, non-negative functions $\{f_\alpha\}$ in $C(K)$ (i.e. $\alpha \leq \beta$ implies $f_\beta \leq f_\beta$, and $\|f_\alpha\| \leq k$ for all α with a fixed k) has a least upper bound in $C(K)$.* Then K is **Stonean**

Remark. The converse is also true (cf. [192]).

Proof. Let G be an open set in K, and let (f_α) be the set of all continuous functions on K such that $0 \leq f_\alpha \leq 1$ and the support of $f_\alpha \subset G$. Then $\{f_\alpha\}$ is a bounded increasing directed set under the order $\leq$.

Put $f = \underset{\alpha}{\text{l.u.b.}} f_\alpha$ in $C(K)$ and $g(t) = \underset{\alpha}{\text{l.u.b.}} f_\alpha(t)$ for $t \in K$. Since $f_\alpha \leq f$, $g(t) \leq f(t)$ for all $t \in K$.

For $t \in G$, there exists an f_α such that $f_\alpha(t) = 1$; hence $g(t) = 1$ and so $f(t) = 1$ on G, since $f \wedge 1 \geq f_\alpha$ for all α, where $(f \wedge 1)(t) = \min(f(t), 1)$. Therefore $f(t) = 1$ on $\bar{G}$, where $\bar{G}$ is the closure of G. Suppose that there exists a $t_0 \varepsilon K - \bar{G}$ such that $f(t_0) > 0$. Take a positive h in $C(K)$ such that $h = 1$ on $\bar{G}$ and $h(t_0) = 0$. Then $f_\alpha \leq f \wedge h < f$, a contradiction. q.e.d.

References. [31], [192].

1.4. Positive Elements of a C^*-Algebra

Let h be a self-adjoint element of the C^*-algebra $\mathscr{A}$. h is said to be positive if $\mathrm{Sp}(h)$ is contained in the non-negative reals. We shall denote this by $h \geq 0$ or $0 \leq h$. Let $C_0(\Omega)$ be the commutative C^*-subalgebra of $\mathscr{A}$ generated by h. Then by 1.2.3, $\Omega \cup (\infty) = \mathrm{Sp}(h) \cup (0) \subset [0, \|h\|]$. Let $C_0([0, \|h\|])$ be the C^*-algebra of all complex valued continuous functions vanishing at 0 on the closed interval $[0, \|h\|]$. For $f \in C_0([0, \|h\|])$, let $f|\mathscr{A}$ be the restriction of f to $\mathrm{Sp}(h) \cup (0)$. Then $f|\mathscr{A} \in \mathscr{A}$. We shall denote $f|\mathscr{A}$ by $f(h)$. Clearly the mapping $\Phi: f \to f(h)$ of $C_0([0, \|h\|])$ into $\mathscr{A}$ is a $*$-homomorphism and $\Phi(C_0([0, \|h\|])) = C_0(\Omega)$, since an arbitrary complex valued continuous function g on the closed subset $\mathrm{Sp}(h) \cup (0)$ of $[0, \|h\|]$, which vanishes at 0, can be extended to an element of $C_0([0, \|h\|])$.

$$f(h) = h \quad \text{if } f(\lambda) = \lambda \quad \text{for } \lambda \in [0, \|h\|].$$

1.4.1. Proposition. *Let h be a positive element in $\mathscr{A}$. Then for an arbitrary positive integer n, there exists a unique positive element k in $\mathscr{A}$ such that $k^n = h$. This unique k is denoted by $\sqrt[n]{h}$ or $h^{1/n}$.*

Proof. In the above considerations, we shall consider a continuous function $f(\lambda) = \sqrt[n]{\lambda}$ on $[0, \|h\|]$. Then $f(h)^n = (f^n)(h) = h$. Now suppose that $k_1^n = h$ for some positive k_1 in $\mathscr{A}$. By the Stone-Weierstrass theorem, the function $\sqrt[n]{\lambda}$ can be uniformly approximated by polynomials p of λ such that $p(0) = 0$; hence $f(h)$ is a uniform limit of polynomials $p(h)$.

$$\text{Since } k_1 h = h k_1, \qquad k_1 f(h) = f(h) k_1.$$

Now let $C_0(\Omega)$ be the commutative C^*-subalgebra of $\mathscr{A}$ generated by k_1 and $f(h)$. Then $k_1(t), f(h)(t) \geq 0$ and $k_1(t)^n = f(h)(t)^n$ for $t \in \Omega$. Hence $k_1(t) = f(h)(t)$ for $t \in \Omega$. q.e.d.

1.4.2. Theorem. *Let P be the set of all positive elements in $\mathscr{A}$. Then P is a convex cone.*

Proof. We may assume that $\mathscr{A}$ has an identity. Let k be a self-adjoint element of $\mathscr{A}$ with $\|k\| \leq 1$.

By representing the commutative C^*-subalgebra of $\mathscr{A}$ generated by 1 and k as $C(K)$, we easily see that k is positive if and only if $\|1 - k\| \leq 1$. Clearly $k \in P$ implies $\lambda k \in P$ $(\lambda \geq 0)$. Let $k_1, k_2 \in P$; then

$$\left\| 1 - \frac{k_1 + k_2}{\|k_1\| + \|k_2\|} \right\| \leq \frac{\|k_1\| \, \|1 - k_1/\|k_1\|\,\| + \|k_2\| \, \|1 - k_2/\|k_2\|\,\|}{\|k_1\| + \|k_2\|} \leq 1.$$

Hence $k_1 + k_2 \in P$. q.e.d.

Let $\mathscr{A}^s$ be the set of all self-adjoint elements in $\mathscr{A}$. By using P we can define a partial ordering $\leq$ in $\mathscr{A}^s$ as follows: $a \leq b$ (or $b \geq a$) if $b - a \in P$.

If $a \in \mathscr{A}^s$, we can easily see, by representing the C^*-subalgebra generated by a as a $C_0(\Omega)$, that a can be written as follows: $a = a_1 - a_2$ with $a_1 \cdot a_2 = 0$ and $a_1, a_2 \geq 0$. Moreover such a decomposition is unique—in fact, let $a = a_1' - a_2'$ where $a_1' \cdot a_2' = 0$ and $a_1', a_2' \geq 0$; then $(a^2)^{\frac{1}{2}} = a_1' + a_2' = a_1 + a_2$. Hence $\dfrac{a + (a^2)^{\frac{1}{2}}}{2} = a_1 = a_1'$ and so $a_2' = a_2$.

1.4.3. Definition. *The above decomposition $a = a_1 - a_2$ with $a_1 \cdot a_2 = 0$ and $a_1, a_2 \geq 0$ is called the orthogonal decomposition of a. a_1 (resp. $-a_2$) is called the positive (resp. negative) part of a, and a_1 (resp. a_2) is denoted by a^+ (resp. a^-). Moreover $a_1 + a_2$ is called the absolute value of a, denoted by $|a|$.*

1.4.4. Theorem. *The following conditions are equivalent:*

1. $h \geq 0$
2. $h = x^* x$ for some $x \in \mathscr{A}$.

Proof. 1. $\Rightarrow$ 2. is clear.
2. $\Rightarrow$ 1. We can assume that $\mathscr{A}$ has an identity.

Suppose $x^* x$ is not positive. Let $C(K)$ be the C^*-subalgebra of $\mathscr{A}$ generated by $x^* x$ and 1. There exists a point t_0 in K such that $x^* x(t_0) < 0$; hence there is a positive element a in $C(K)$ such that $a x^* x a < 0$. Let $x a = h_1 + i h_2$ $(h_1, h_2 \in \mathscr{A}^s)$. Then

$$(x a)^* (x a) = h_1^2 + h_2^2 + i h_1 h_2 - i h_2 h_1 \quad \text{and}$$
$$(x a)(x a)^* = h_1^2 + h_2^2 - i h_1 h_2 + i h_2 h_1 .$$

Hence $(x a)^* (x a) + (x a)(x a)^* = 2(h_1^2 + h_2^2) > 0$, and so

$$(x a)(x a)^* = -(x a)^* (x a) + 2(h_1^2 + h_2^2) > 0.$$

On the other hand, by 1.1.8, $\mathrm{Sp}((x a)^*(x a)) \cup (0) = \mathrm{Sp}((x a)(x a)^*) \cup (0)$; hence $(x a)(x a)^* \leq 0$, a contradiction. q.e.d.

1.4.5. Proposition. *Let $\mathscr{A}$ be a C^*-algebra with identity. Then every element of $\mathscr{A}$ is a finite linear combination of unitary elements.*

Proof. Let h be a positive element in $\mathscr{A}$; then $\dfrac{h}{\|h\|} \pm i\sqrt{1 - h^2/\|h\|^2}$ are unitary elements. Hence h is a finite linear combination of unitary elements. q.e.d.

Concluding remarks on 1.4.

Theorem 1.4.2 is due to Fukamiya [52] and Kelley and Vaught [101]. Theorem 1.4.4 is due to Kaplansky [175]. These results are the key lemmas which gave the positive answer to the non-commutative case of Gelfand-Naimark's query after a decade of mystery (cf. Concluding remarks in 1.16).

References. [52], [101].

1.5. Positive Linear Functionals on a C^*-Algebra

Let f be a linear functional on a C^*-algebra $\mathscr{A}$. Define $f^*(x)=\overline{f(x^*)}$ for $x\in\mathscr{A}$. Then f^* is also a linear functional on $\mathscr{A}$. f^* is called the adjoint of f. If $f^*=f$, we call f self-adjoint.

Every linear functional on $\mathscr{A}$ can be represented in the form $f=f_1+if_2$, where f_1,f_2 are self-adjoint. In fact, $f_1=1/2(f+f^*)$, $f_2=1/2i(f-f^*)$.

A linear functional φ on $\mathscr{A}$ is called positive if $\varphi(x^*x)\geq 0$ for $x\in\mathscr{A}$. φ is automatically self-adjoint.

We shall denote the positivity of φ by $\varphi\geq 0$ (or $0\leq\varphi$). For two self-adjoint linear functionals f_1,f_2, we shall denote $f_2-f_1\geq 0$ by $f_2\geq f_1$ (or $f_1\leq f_2$).

A positive linear functional φ satisfies the Schwartz inequality: $|\varphi(y^*x)|^2\leq\varphi(x^*x)\varphi(y^*y)$ for $x,y\in\mathscr{A}$.

1.5.1. Proposition. *Let $\mathscr{A}$ be a C^*-algebra with identity. Then a positive linear functional φ is bounded and $\|\varphi\|=\varphi(1)$.*

Proof. $|\varphi(x)|\leq\varphi(1)^{\frac{1}{2}}\varphi(x^*x)^{\frac{1}{2}}$, and $\|x^*x\|1-x^*x\geq 0$; hence $\varphi(x^*x)\leq\|x^*x\|\varphi(1)$ and so $|\varphi(x)|\leq\varphi(1)\|x\|$. q.e.d.

1.5.2. Proposition. *Let $\mathscr{A}$ be a C^*-algebra, and let f be a bounded linear functional on $\mathscr{A}$ such that $f(h)=\|f\|\,\|h\|$ for some positive $h(>0)\in\mathscr{A}$. Then f is a positive linear functional.*

Proof. By the Hahn-Banach theorem we can assume that $\mathscr{A}$ has an identity. Put $f_0=\dfrac{f}{\|f\|}$ and $h_0=\dfrac{h}{\|h\|}$. Then $f_0(h_0)=1$. First of all we shall show $f_0(1)=$real. Suppose $f_0(1)=\alpha+i\beta$ (α,β reals, $\beta\neq 0$). Then for an arbitrary real λ, we have

$$|f_0(1+\lambda ih_0)|=|\alpha+i(\lambda+\beta)|\geq|\lambda+\beta|.$$

On the other hand,

$$|f_0(1+\lambda ih_0)|\leq\|1+\lambda ih_0\|\leq(1+\lambda^2)^{\frac{1}{2}}.$$

Hence

$$|\lambda+\beta|\leq(1+\lambda^2)^{\frac{1}{2}}\quad\text{for all real }\lambda,$$

and so

$$\lambda^2+2\lambda\beta+\beta^2\leq 1+\lambda^2\quad\text{for all real }\lambda.$$