天元基金 影印系列丛书

Allen Hatcher 著

代数拓扑

ALGEBRAIC TOPOLOGY

清华大学出版社

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Allen Hatcher is Professor of Mathematics at Cornell University.

Preface

This book was written to be a readable introduction to algebraic topology with rather broad coverage of the subject. The viewpoint is quite classical in spirit, and stays well within the confines of pure algebraic topology. In a sense, the book could have been written thirty or forty years ago since virtually everything in it is at least that old. However, the passage of the intervening years has helped clarify what are the most important results and techniques. For example, CW complexes have proved over time to be the most natural class of spaces for algebraic topology, so they are emphasized here much more than in the books of an earlier generation. This emphasis also illustrates the book's general slant towards geometric, rather than algebraic, aspects of the subject. The geometry of algebraic topology is so pretty, it would seem a pity to slight it and to miss all the intuition it provides.

At the elementary level, algebraic topology separates naturally into the two broad channels of homology and homotopy. This material is here divided into four chapters, roughly according to increasing sophistication, with homotopy split between Chapters 1 and 4, and homology and its mirror variant cohomology in Chapters 2 and 3. These four chapters do not have to be read in this order, however. One could begin with homology and perhaps continue with cohomology before turning to homotopy. In the other direction, one could postpone homology and cohomology until after parts of Chapter 4. If this latter strategy is pushed to its natural limit, homology and cohomology can be developed just as branches of homotopy theory. Appealing as this approach is from a strictly logical point of view, it places more demands on the reader, and since readability is one of the first priorities of the book, this homotopic interpretation of homology and cohomology is described only after the latter theories have been developed independently of homotopy theory.

Preceding the four main chapters there is a preliminary Chapter 0 introducing some of the basic geometric concepts and constructions that play a central role in both the homological and homotopical sides of the subject. This can either be read before the other chapters or skipped and referred back to later for specific topics as they become needed in the subsequent chapters.

Each of the four main chapters concludes with a selection of additional topics that the reader can sample at will, independent of the basic core of the book contained in the earlier parts of the chapters. Many of these extra topics are in fact rather important in the overall scheme of algebraic topology, though they might not fit into the time constraints of a first course. Altogether, these additional topics amount to nearly half the book, and they are included here both to make the book more comprehensive and to give the reader who takes the time to delve into them a more substantial sample of the true richness and beauty of the subject.

Not included in this book is the important but somewhat more sophisticated topic of spectral sequences. It was very tempting to include something about this marvelous tool here, but spectral sequences are such a big topic that it seemed best to start with them afresh in a new volume. This is tentatively titled 'Spectral Sequences in Algebraic Topology' and is referred to herein as [SSAT]. There is also a third book in progress, on vector bundles, characteristic classes, and K-theory, which will be largely independent of [SSAT] and also of much of the present book. This is referred to as [VBKT], its provisional title being 'Vector Bundles and K-Theory.'

In terms of prerequisites, the present book assumes the reader has some familiarity with the content of the standard undergraduate courses in algebra and point-set topology. In particular, the reader should know about quotient spaces, or identification spaces as they are sometimes called, which are quite important for algebraic topology. Good sources for this concept are the textbooks [Armstrong 1983] and [Jänich 1984] listed in the Bibliography.

A book such as this one, whose aim is to present classical material from a rather classical viewpoint, is not the place to indulge in wild innovation. There is, however, one small novelty in the exposition that may be worth commenting upon, even though in the book as a whole it plays a relatively minor role. This is the use of what we call Δ-complexes, which are a mild generalization of the classical notion of a simplicial complex. The idea is to decompose a space into simplices allowing different faces of a simplex to coincide and dropping the requirement that simplices are uniquely determined by their vertices. For example, if one takes the standard picture of the torus as a square with opposite edges identified and divides the square into two triangles by cutting along a diagonal, then the result is a Δ-complex structure on the torus having 2 triangles, 3 edges, and 1 vertex. By contrast, a simplicial complex structure on the torus must have at least 14 triangles, 21 edges, and 7 vertices. So Δ-complexes provide a significant improvement in efficiency, which is nice from a pedagogical viewpoint since it cuts down on tedious calculations in examples. A more fundamental reason for considering Δ-complexes is that they seem to be very natural objects from the viewpoint of algebraic topology. They are the natural domain of definition for simplicial homology, and a number of standard constructions produce Δ-complexes rather than simplicial complexes, for instance the singular complex of a space, or the classifying space of a discrete group or category. Historically, Δ-complexes were first introduced by Eilenberg and Zilber in 1950 under the name of semisimplicial complexes. This term later came to mean something different, however, and the original notion seems to have been largely ignored since.

This book will remain available online in electronic form after it has been printed in the traditional fashion. The web address is

http://www.math.cornell.edu/~hatcher

One can also find here the parts of the other two books in the sequence that are currently available. Although the present book has gone through countless revisions, including the correction of many small errors both typographical and mathematical found by careful readers of earlier versions, it is inevitable that some errors remain, so the web page will include a list of corrections to the printed version. With the electronic version of the book it will be possible not only to incorporate corrections but also to make more substantial revisions and additions. Readers are encouraged to send comments and suggestions as well as corrections to the email address posted on the web page.

Standard Notations

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$: the integers, rationals, reals, complexes, quaternions, and Cayley octonions

$\mathbb{Z}_n$: the integers mod n

$\mathbb{R}^n$: n-dimensional Euclidean space

$\mathbb{C}^n$: complex n-space

$I = [0,1]$: the unit interval

S^n: the unit sphere in $\mathbb{R}^{n+1}$, all vectors of length 1

D^n: the unit disk or ball in $\mathbb{R}^n$, all vectors of length ≤ 1

$\partial D^n = S^{n-1}$: the boundary of the n-disk

$\mathbb{1}$: the identity function from a set to itself

$\amalg$: disjoint union of sets or spaces

$\times$, $\prod$: product of sets, groups, or spaces

$\approx$: isomorphism

$A \subset B$ or $B \supset A$: set-theoretic containment, not necessarily proper

iff: if and only if

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Chapter 0

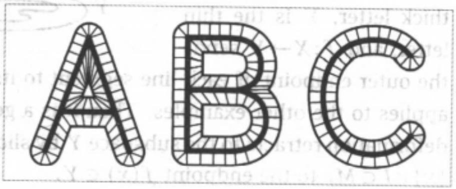
Some Underlying Geometric Notions

The aim of this short preliminary chapter is to introduce a few of the most common geometric concepts and constructions in algebraic topology. The exposition is somewhat informal, with no theorems or proofs until the last couple pages, and it should be read in this informal spirit, skipping bits here and there. In fact, this whole chapter could be skipped now, to be referred back to later for basic definitions.

To avoid overusing the word 'continuous' we adopt the convention that maps between spaces are always assumed to be continuous unless otherwise stated.

Homotopy and Homotopy Type

One of the main ideas of algebraic topology is to consider two spaces to be equivalent if they have 'the same shape' in a sense that is much broader than homeomorphism. To take an everyday example, the letters of the alphabet can be written either as unions of finitely many straight and curved line segments, or in thickened forms that are compact subsurfaces of the plane bounded by simple closed curves. In each case the thin letter is a subspace of the thick letter, and we can continuously shrink the thick letter to the thin one. A nice way to do this is to decompose a thick letter, call it $\mathbf{X}$, into line segments connecting each point on the outer boundary of $\mathbf{X}$ to a unique point of the thin subletter X, as indicated in the figure. Then we can shrink $\mathbf{X}$ to X by sliding each point of $\mathbf{X} - X$ into X along the line segment that contains it. Points that are already in X do not move.

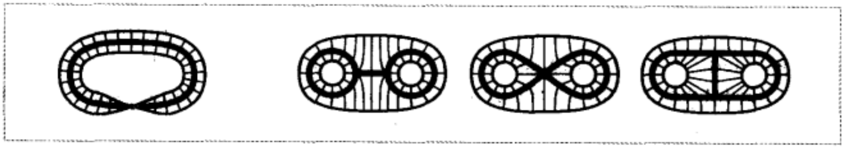


We can think of this shrinking process as taking place during a time interval $0 \le t \le 1$, and then it defines a family of functions $f_t : \mathbf{X} \to \mathbf{X}$ parametrized by $t \in I = [0,1]$, where $f_t(x)$ is the point to which a given point $x \in \mathbf{X}$ has moved at time t.

Naturally we would like $f_t(x)$ to depend continuously on both t and x, and this will be true if we have each $x \in \mathbf{X} - \mathsf{X}$ move along its line segment at constant speed so as to reach its image point in X at time $t = 1$, while points $x \in \mathsf{X}$ are stationary, as remarked earlier.

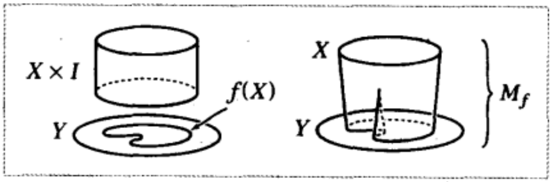
Examples of this sort lead to the following general definition. A **deformation retraction** of a space X onto a subspace A is a family of maps $f_t : X \to X$, $t \in I$, such that $f_0 = \mathbb{1}$ (the identity map), $f_1(X) = A$, and $f_t | A = \mathbb{1}$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x,t) \mapsto f_t(x)$, is continuous.

It is easy to produce many more examples similar to the letter examples, with the deformation retraction f_t obtained by sliding along line segments. The figure on the left below shows such a deformation retraction of a Möbius band onto its core circle.



The three figures on the right show deformation retractions in which a disk with two smaller open subdisks removed shrinks to three different subspaces.

In all these examples the structure that gives rise to the deformation retraction can be described by means of the following definition. For a map $f : X \to Y$, the **mapping cylinder** M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by identifying each $(x,1) \in X \times I$ with $f(x) \in Y$. In the letter examples, the space X is the outer boundary of the thick letter, Y is the thin letter, and $f : X \to Y$ sends the outer endpoint of each line segment to its inner endpoint. A similar description applies to the other examples. Then it is a general fact that a mapping cylinder M_f deformation retracts to the subspace Y by sliding each point (x,t) along the segment $\{x\} \times I \subset M_f$ to the endpoint $f(x) \in Y$.



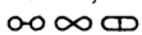
Not all deformation retractions arise in this way from mapping cylinders, however. For example, the thick $\mathbf{X}$ deformation retracts to the thin X, which in turn deformation retracts to the point of intersection of its two crossbars. The net result is a deformation retraction of $\mathbf{X}$ onto a point, during which certain pairs of points follow paths that merge before reaching their final destination. Later in this section we will describe a considerably more complicated example, the so-called 'house with two rooms,' where a deformation retraction to a point can be constructed abstractly, but seeing the deformation with the naked eye is a real challenge.

A deformation retraction $f_t : X \to X$ is a special case of the general notion of a **homotopy**, which is simply any family of maps $f_t : X \to Y$, $t \in I$, such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous. One says that two maps $f_0, f_1 : X \to Y$ are **homotopic** if there exists a homotopy f_t connecting them, and one writes $f_0 \simeq f_1$.

In these terms, a deformation retraction of X onto a subspace A is a homotopy from the identity map of X to a **retraction** of X onto A, a map $r : X \to X$ such that $r(X) = A$ and $r|A = \mathbb{1}$. One could equally well regard a retraction as a map $X \to A$ restricting to the identity on the subspace $A \subset X$. From a more formal viewpoint a retraction is a map $r : X \to X$ with $r^2 = r$, since this equation says exactly that r is the identity on its image. Retractions are the topological analogs of projection operators in other parts of mathematics.

Not all retractions come from deformation retractions. For example, every space X retracts onto any point $x_0 \in X$ via the map sending all of X to x_0. But a space that deformation retracts onto a point must certainly be path-connected, since a deformation retraction of X to a point x_0 gives a path joining each $x \in X$ to x_0. It is less trivial to show that there are path-connected spaces that do not deformation retract onto a point. One would expect this to be the case for the letters 'with holes,' A, B, D, O, P, Q, R. In Chapter 1 we will develop techniques to prove this.

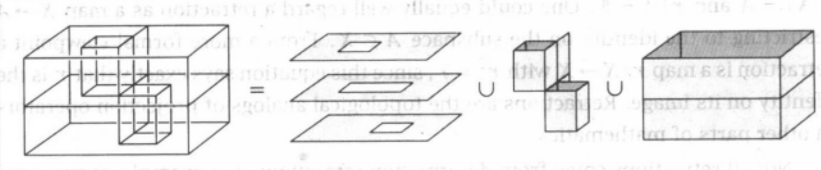
A homotopy $f_t : X \to X$ that gives a deformation retraction of X onto a subspace A has the property that $f_t|A = \mathbb{1}$ for all t. In general, a homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t is called a **homotopy relative to** A, or more concisely, a homotopy rel A. Thus, a deformation retraction of X onto A is a homotopy rel A from the identity map of X to a retraction of X onto A.

If a space X deformation retracts onto a subspace A via $f_t : X \to X$, then if $r : X \to A$ denotes the resulting retraction and $i : A \to X$ the inclusion, we have $ri = \mathbb{1}$ and $ir \simeq \mathbb{1}$, the latter homotopy being given by f_t. Generalizing this situation, a map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $fg \simeq \mathbb{1}$ and $gf \simeq \mathbb{1}$. The spaces X and Y are said to be **homotopy equivalent** or to have the same **homotopy type**. The notation is $X \simeq Y$. It is an easy exercise to check that this is an equivalence relation, in contrast with the nonsymmetric notion of deformation retraction. For example, the three graphs  are all homotopy equivalent since they are deformation retracts of the same space, as we saw earlier, but none of the three is a deformation retract of any other.

It is true in general that two spaces X and Y are homotopy equivalent if and only if there exists a third space Z containing both X and Y as deformation retracts. For the less trivial implication one can in fact take Z to be the mapping cylinder M_f of any homotopy equivalence $f : X \to Y$. We observed previously that M_f deformation retracts to Y, so what needs to be proved is that M_f also deformation retracts to its other end X if f is a homotopy equivalence. This is shown in Corollary 0.21.

A space having the homotopy type of a point is called **contractible**. This amounts to requiring that the identity map of the space be **nullhomotopic**, that is, homotopic to a constant map. In general, this is slightly weaker than saying the space deformation retracts to a point; see the exercises at the end of the chapter for an example distinguishing these two notions.

Let us describe now an example of a 2-dimensional subspace of $\mathbb{R}^3$, known as the *house with two rooms*, which is contractible but not in any obvious way. To build this



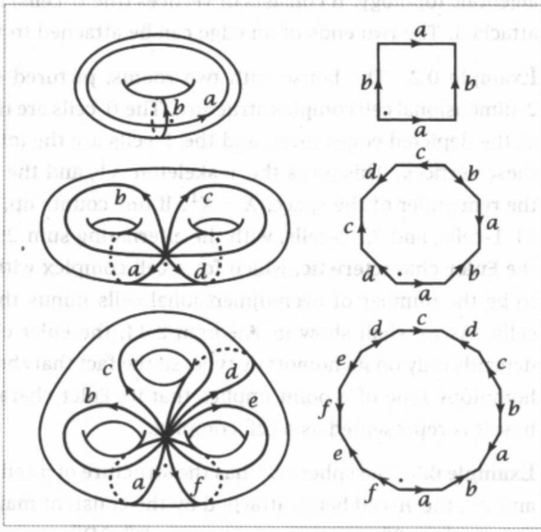
space, start with a box divided into two chambers by a horizontal rectangle, where by a 'rectangle' we mean not just the four edges of a rectangle but also its interior. Access to the two chambers from outside the box is provided by two vertical tunnels. The upper tunnel is made by punching out a square from the top of the box and another square directly below it from the middle horizontal rectangle, then inserting four vertical rectangles, the walls of the tunnel. This tunnel allows entry to the lower chamber from outside the box. The lower tunnel is formed in similar fashion, providing entry to the upper chamber. Finally, two vertical rectangles are inserted to form 'support walls' for the two tunnels. The resulting space X thus consists of three horizontal pieces homeomorphic to annuli plus all the vertical rectangles that form the walls of the two chambers.

To see that X is contractible, consider a closed ε-neighborhood $N(X)$ of X. This clearly deformation retracts onto X if ε is sufficiently small. In fact, $N(X)$ is the mapping cylinder of a map from the boundary surface of $N(X)$ to X. Less obvious is the fact that $N(X)$ is homeomorphic to D^3, the unit ball in $\mathbb{R}^3$. To see this, imagine forming $N(X)$ from a ball of clay by pushing a finger into the ball to create the upper tunnel, then gradually hollowing out the lower chamber, and similarly pushing a finger in to create the lower tunnel and hollowing out the upper chamber. Mathematically, this process gives a family of embeddings $h_t : D^3 \to \mathbb{R}^3$ starting with the usual inclusion $D^3 \hookrightarrow \mathbb{R}^3$ and ending with a homeomorphism onto $N(X)$.

Thus we have $X \simeq N(X) = D^3 \simeq point$, so X is contractible since homotopy equivalence is an equivalence relation. In fact, X deformation retracts to a point. For if f_t is a deformation retraction of the ball $N(X)$ to a point $x_0 \in X$ and if $r : N(X) \to X$ is a retraction, for example the end result of a deformation retraction of $N(X)$ to X, then the restriction of the composition rf_t to X is a deformation retraction of X to x_0. However, it is quite a challenging exercise to see exactly what this deformation retraction looks like.

Cell Complexes

A familiar way of constructing the torus $S^1 \times S^1$ is by identifying opposite sides of a square. More generally, an orientable surface M_g of genus g can be constructed from a polygon with $4g$ sides by identifying pairs of edges, as shown in the figure in the first three cases $g = 1, 2, 3$. The $4g$ edges of the polygon become a union of $2g$ circles in the surface, all intersecting in a single point. The interior of the polygon can be thought of as an open disk, or a **2-cell**, attached to the union of the $2g$ circles. One can also regard the union of the circles as being obtained from their common point of intersection, by attaching $2g$ open arcs, or **1-cells**. Thus the surface can be built up in stages: Start with a point, attach 1-cells to this point, then attach a 2-cell.



A natural generalization of this is to construct a space by the following procedure:

(1) Start with a discrete set X^0, whose points are regarded as 0-cells.

(2) Inductively, form the ***n*-skeleton** X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$ of X^{n-1} with a collection of n-disks D^n_α under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$. Thus as a set, $X^n = X^{n-1} \coprod_\alpha e^n_\alpha$ where each e^n_α is an open n-disk.

(3) One can either stop this inductive process at a finite stage, setting $X = X^n$ for some $n < \infty$, or one can continue indefinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

A space X constructed in this way is called a **cell complex** or **CW complex**. The explanation of the letters 'CW' is given in the Appendix, where a number of basic topological properties of cell complexes are proved. The reader who wonders about various point-set topological questions lurking in the background of the following discussion should consult the Appendix for details.