CNC Series on Complexity, Nonlinearity and Chaos – Vol. 4

ASYMPTOTIC INTEGRATION AND STABILITY

For Ordinary, Functional and Discrete Differential Equations of Fractional Order

Dumitru Baleanu
Octavian G. Mustafa

World Scientific

CNC Series on Complexity, Nonlinearity and Chaos – Vol. 4

ASYMPTOTIC INTEGRATION AND STABILITY

For Ordinary, Functional and Discrete Differential Equations of Fractional Order

Dumitru Baleanu
Institute of Space Sciences, Romania & Çankaya University, Turkey

Octavian G. Mustafa
University of Craiova, Romania

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Baleanu, D. (Dumitru)
Asymptotic integration and stability : for ordinary, functional and discrete differential equations of fractional order / Dumitru Baleanu, Çankaya University, Ankara, Turkey, Institute of Space Sciences, Bucharest, Romania, Octavian G. Mustafa, University of Craiova, Romania.
pages cm. -- (Series on complexity, nonlinearity and chaos ; volume 4)
Includes bibliographical references and index.
ISBN 978-9814641098 (hardcover : alk. paper)
1. Fractional calculus. 2. Fractional differential equations. I. Mustafa, Octavian G. II. Title.
QA314.B35 2015
515'.83--dc23

2014045570

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed in Singapore

ASYMPTOTIC INTEGRATION AND STABILITY

For Ordinary, Functional and Discrete Differential Equations of Fractional Order

Series on Complexity, Nonlinearity and Chaos

ISSN 2010-0019

Series Editor: Albert C.J. Luo
(Southern Illinois University Edwardsville, USA)

Aims and Scope

The books in this series will focus on the recent developments, findings and progress on fundamental theories and principles, analytical and symbolic approaches, computational techniques in nonlinear physical science and nonlinear mathematics.

Topics of interest in Complexity, Nonlinearity and Chaos include but not limited to:

- New findings and discoveries in nonlinear physics and mathematics,
- Complexity and mathematical structures in nonlinear physics,
- Nonlinear phenomena and observations in nature,
- Computational methods and simulations in complex systems,
- New theories, and principles and mathematical methods,
- Stability, bifurcation, chaos and fractals in nonlinear physical science.

Vol. 1 Ray and Wave Chaos in Ocean Acoustics: Chaos in Waveguides
D. Makarov, S. Prants, A. Virovlyansky & G. Zaslavsky

Vol. 2 Applications of Lie Group Analysis in Geophysical Fluid Dynamics
N. H. Ibragimov & R. N. Ibragimov

Vol. 3 Fractional Calculus: Models and Numerical Methods
D. Baleanu, K. Diethelm & J. J. Trujillo

Vol. 4 Asymptotic Integration and Stability: For Ordinary, Functional and Discrete Differential Equations of Fractional Order
D. Baleanu & O. G. Mustafa

To our wives

Mihaela-Cristina and Dorina

Preface

The aim of the fractional calculus is to study the fractional order integral and derivative operator over real and complex domains as well as their applications. We recall that the tools of fractional calculus are as old as calculus itself.

Nowadays there are strong motivations (the unelucidated nature of the dark matter and dark energy, the difficult reconciliation of Einstein's General Relativity (GR) and Quantum Theory) to consider alternative theories that modify, extend or replace GR. We recall that some of these theories presume a higher dimensional space-time, and part of them predict violations of the physics fundamental principles: the Equivalence Principle and Lorentz symmetry could be broken, the fundamental constants could vary, the space could be anisotropic, and the physics could become nonlocal.

Fractional calculus becomes very powerful in the study of the anomalous social and physical behaviors, where scaling power law of fractional order appears universal as an empirical description of the complex phenomena.

The classical mathematical models, including nonlinear models, do not give adequate results in many cases where power law is clearly present.

During the last years, the asymptotic integration and the stability of fractional differential equations become an important research topic in the field of fractional calculus and its applications. Thus, a better understanding of these concepts represents one of the major tasks for researchers working on these fields and related topics.

The book contains eleven chapters and it is based mainly on the results reported by the authors during the last few years. The first chapter is about the differential operators of order $1+\alpha$ and their integral counterpart. The second chapter describes the existence and the uniqueness of solution for the differential equations of order α. Chapter three debates the position of the zeros, the Bihari inequality and the asymptotic behaviour of solutions for the differential equations of order α. Chapter four describes the asymptotic integration for the differential equations of order $1 + \alpha$. In chapter five we present the existence and the uniqueness of solutions for some delay differential equations within Caputo derivative. In chapter six we discuss the existence and the positive solutions for some delay fractional differential equations with generalized N term. The stability of a class of discrete fractional nonautonomous systems is shown in chapter seven. Mittag-Leffler stability theorem for fractional nonlinear systems with delay is the subject of the chapter eight. Chapter nine is concentrated on the Razumikhin stability theorem for some fractional systems with delay. Chapter ten deals with the controlability of some fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. Finally, the book ends with the approximate controllability of Sobolev type nonlocal fractional stochastic dynamic systems in Hilbert spaces.

We would like to thank to all of our co-authors who helped us in writing this book by providing many interesting comments and remarks.

Finally, we deeply thank the editorial assistance of World Scientific Publishing Co., especially Ms. L.F. Kwong and Mr. Rajesh Babu.

Dumitru Baleanu
Çankaya University, Ankara, Turkey
Institute of Space Sciences, Bucharest, Romania

Octavian Mustafa
University of Craiova, Romania

Contents

Chapter 1

The Differential Operators of Order $1 + \alpha$ and Their Integral Counterparts

When asked about the significance of the *non-integer order* for a differential operator by anyone without special interest in the mathematics of differential equations, the *fractionist*[1] can provide the inquirer with a simple analogy, described in the following.

Take a continuous function $f : I \to \mathbb{R}$, where $I = [a, b]$ is an interval of real numbers, and write down the identity below

$$\begin{aligned} f(t) &= \frac{d}{dt}\left[\int_a^t f(s)ds\right] \\ &= (\text{Diff} \circ \text{Int})(f)(t) = \mathcal{E}(f)(t), \quad t \in (a, b). \end{aligned}$$

The identity expresses the fact that you did one integration — the order of the expression $\mathcal{E}(f) = \text{Int}(f)$ is now $+1$ — followed by one differentiation — the new, and final, order of $\mathcal{E}$ is $(+1) + (-1) = 0$ —.

Let's go further and recall, via integration by parts, that

$$f(t) = \frac{1}{(n-1)!} \cdot \frac{d^n}{dt^n}\left[\int_a^t (t-s)^{n-1} f(s)ds\right], \quad t \in (a, b). \tag{1.1}$$

As before, the integral brings $+1$ into the sum (for computing the order), while the "$t - s$" component is responsible for another $n - 1$ *units* of order. The final order of $\mathcal{E}$ is $(+1) + (n-1) \cdot (+1) + n \cdot (-1) = 0$.

Now, given $z \in \mathbb{C}$ and $t, \varepsilon > 0$, the mapping $z \mapsto t^z = e^{z \cdot \log t}$ is entire and it makes sense to wonder about the function $z \mapsto \int_0^{t+\varepsilon} (t-s)^z f(s)ds$ being holomorphic [Hille (1959), pp. 72, 230], a property sometimes called upon as analytic [Ablowitz and Fokas (2003), p. 24]. Using the technique from e.g., [Rudin (1987), Chap. 10, Ex. 16], it can be established that the latter function is, in fact, entire. In particular, taking $\alpha \in \mathbb{R} \backslash \mathbb{N}$ and $n \in \mathbb{N}$,

[1] That is, the analyst of the "fractionals".

with $n \geq 1$, the quantity

$$\mathcal{E}(f)(t) = \frac{d^n}{dt^n}\left[\int_a^{t+\varepsilon}(t-s)^{\alpha}f(s)ds\right], \quad t \in (a,b),$$

makes sense. Without paying any attention[2] to the extra "ε", deduce that the order of the expression must be $(+1)+\alpha\cdot(+1)+n\cdot(-1)=\alpha-n+1$.

To conclude our analogy, we get rid of ε by making it tend to zero and look for some positive constant to mimic the $\frac{1}{(n-1)!}$ coefficient from (1.1). The formula of a new derivative having a *non-integer* order "close" to n is expressed as

$$c_n \cdot \frac{d^n}{dt^n}\left[\int_a^{t}(t-s)^{\alpha_n}f(s)ds\right], \quad t \in (a,b), \tag{1.2}$$

where $\alpha_n \in (-1,0]$ and $c_n > 0$. It resembles the classical Riemann-Liouville construction [Podlubny (1999a), p. 68].

1.1 The Gamma Function

For a proper constant c_n, we rely on Euler's integral (of second kind) Γ. This function can be stated as

$$\Gamma(z) = \int_0^{+\infty} \mathrm{e}^{-t}t^{z-1}dt, \quad \text{where } t^{\xi} = \begin{cases} 0, & t=0, \\ \mathrm{e}^{\xi\cdot\log t}, & t>0, \end{cases} \quad \xi, z \in \mathbb{C},$$

and the real part of z is positive [Olver (1974), p. 31].

Since we shall employ everywhere only the most common of its properties, we remind the interested reader that he or she can delve anytime into the detailed proofs of results about the Gamma function from [Olver (1974); Podlubny (1999a)]. As for ourselves, just remember that

$$\begin{aligned} &\Gamma(z+1) = z\cdot\Gamma(z), && \\ &\Gamma(n) = (n-1)!, && n \in \mathbb{N}\backslash\{0\}, \\ &\Gamma(z)\cdot\Gamma(1-z) = \tfrac{\pi}{\sin \pi z}, && \text{when } z \text{ is non-integer}, \\ &\tfrac{\Gamma(z_1)\cdot\Gamma(z_2)}{\Gamma(z_1+z_2)} = \textstyle\int_0^1 v^{z_1-1}(1-v)^{z_2-1}dv, \ z_1, z_2 \in \mathbb{C}, && \end{aligned} \tag{1.3}$$

where the real parts of z_1, z_2 are positive. The last quantity in (1.3) is the Euler integral of first kind, also referred to as the Beta function $B(z_1, z_2)$ [Olver (1974), p. 37].

Following [Baleanu et al. (2011c)], we claim as well that

$$\Gamma(\beta) > \frac{1}{\Gamma(2-\beta)}, \quad \beta \in (0,1). \tag{1.4}$$

[2] For a technical approach, see [Stein (1970), p. 77, Lemma].

To prove the assertion, just remark that

$$\Gamma(\beta)\cdot\Gamma(2-\beta) = (1-\beta)\Gamma(1-\beta)\cdot\Gamma(\beta) = (1-\beta)\cdot\frac{\pi}{\sin(\pi\beta)}$$
$$= \frac{\pi(1-\beta)}{\sin(\pi(1-\beta))},$$

which leads to

$$\lim_{\beta\searrow 0}\Gamma(\beta)\cdot\Gamma(2-\beta) = +\infty, \quad \lim_{\beta\nearrow 1}\Gamma(\beta)\cdot\Gamma(2-\beta) = 1.$$

Observe also that there are no critical points of the function $\beta \mapsto \frac{\pi(1-\beta)}{\sin(\pi\beta)}$ in $(0,1)$. This follows from the fact that the unique solution of the transcendental equation $\tan\pi(1-\beta) = \pi(1-\beta)$ in $[0,1]$ is $\beta^\star = 1$. So,

$$\Gamma(\beta)\cdot\Gamma(2-\beta) > 1, \quad \beta\in(0,1).$$

The claim is established.

1.2 The Riemann-Liouville Derivative

A multitude of notations has been designed to capture the complexity of fractional differentiation and integration, see the monographs [Samko et al. (1993); Miller and Ross (1993); Kilbas et al. (2006)] or the classical treatise [Hille and Phillips (1957), pp. 664, 673]. In a simplified version, the *Riemann-Liouville derivative*, of *order* $\alpha\in(0,1)$, reads as below

$$_0D_t^\alpha(f)(t) = \frac{1}{\Gamma(1-\alpha)}\cdot\frac{d}{dt}\left[\int_0^t \frac{f(s)}{(t-s)^\alpha}ds\right], \quad t>0. \tag{1.5}$$

The subscripts 0 and t in this symbol of derivative[3] hint at the integration interval $(0,t)$, recall (1.2).

The first issue regarding (1.5) is about the existence of the integral inside. As the mapping $\eta_{t,\alpha}$ given by $s\mapsto(t-s)^{-\alpha}$ is a member of $L^1((0,t),\mathbb{R})$, it is natural to ask that $f\in L^\infty((0,t),\mathbb{R})$, see [Rudin (1987), Chap. 3, Th. 3.8]. In particular, this qualifies all the functions from $C([0,t],\mathbb{R})$ as candidates for f.

Since any absolutely integrable function — with respect to the Lebesgue measure on $[0,t]$ — may take infinite values on a null-measure set, we see that $f\eta_{t,\alpha}$ might be infinite in other points beside $s=t$. To take advantage of this remark, introduce $\mathcal{RL}^\beta = \mathcal{RL}^\beta((0,+\infty),\mathbb{R})$ the real linear space of

[3]Given the linearity of the right-hand part of (1.5), we shall use freely either of symbols $_0D_t^\alpha(f)$, $_0D_t^\alpha f$ when referring to the Riemann-Liouville derivative of f.

all the functions $f \in C((0,+\infty),\mathbb{R})$ with $\lim\limits_{t\searrow 0}[t^\beta f(t)] \in \mathbb{R}$ for some $\beta \in [0,1)$. Now,

$$\int_0^t \frac{f(s)}{(t-s)^\alpha}ds = \int_0^t \frac{s^\beta \cdot f(s)}{s^\beta(t-s)^\alpha}ds \tag{1.6}$$
$$= \left[\int_0^{+\infty} \frac{\chi_{(0,\frac{t}{2})}}{s^\beta(t-s)^\alpha} + \int_0^{+\infty} \frac{\chi_{(\frac{t}{2},t)}}{s^\beta(t-s)^\alpha}\right] \cdot s^\beta f(s)\,ds.$$

Again, the ratios inside the integrals are L^1-functions while $s \mapsto s^\beta f(s)$ is in[4] $L^\infty_{\text{loc}}([0,+\infty),\mathbb{R})$ for any $f \in \mathcal{RL}^\beta$. Here, as usual, χ designates the characteristic function of a Lebesgue-measurable set. From now on we shall presume that *every function f involved in a Riemann-Liouville differentiation is a member of some $\mathcal{RL}^\beta$.*

The second issue concerning (1.5) is the differentiability of the integral. To find some reasonable restrictions on f that will lead to this differentiability, notice that we can perform the change of variables $s = t \cdot v$ for the integral inside (1.5), that is

$$\int_0^t \frac{f(s)}{(t-s)^\alpha}ds = t^{1-\alpha} \cdot \int_0^1 \frac{f(tv)}{(1-v)^\alpha}dv, \tag{1.7}$$

see [Rudin (1987), Chap. 7, Th. 7.26].

Assume now that the function f from (1.7) is (locally) absolutely continuous in $[0,+\infty)$, which means it is differentiable *almost everywhere* — we shall use the shorthand notation **a.e.** — and $f' \in L^1_{\text{loc}}([0,+\infty),\mathbb{R})$ [Rudin (1987), Chap. 7, Th. 7.20]. Moreover, we ask that the mapping $s \mapsto s^{1+\beta}f'(s)$ be in $L^\infty_{\text{loc}}([0,+\infty),\mathbb{R})$ for some $\beta \in [0,1)$. We shall refer to this mapping as $s^{1+\beta}f'$ in the following computation. A significant particular case of our restriction is when $f' \in L^\infty_{\text{loc}}([0,+\infty),\mathbb{R})$, which makes f a *locally Lipschitz function.* We recall Rademacher's theorem [Evans and Gariepy (1992), p. 81] which deals with the a.e. differentiability of such functions. Another important situation is when $f \in C^1((0,+\infty),\mathbb{R}) \cap \mathcal{RL}^\gamma$ and $f' \in \mathcal{RL}^{1+\beta}$ for some γ, $\beta \in [0,1)$. These restrictions have been modeled to ensure the existence of the right-hand part of (1.11).

Allow us to recapitulate at this point several elementary facts. First, given $\beta \in (0,1)$ and $x \in (0,1]$, since $\ln x \le 0$, we have $\beta \ln x \ge \ln x$ and, by exponentiation, $x^\beta \ge x$. Second, given $\alpha \in (0,1)$, we get

$$1 = (1-\alpha) + \alpha \le (1-\alpha)^\beta + \alpha^\beta.$$

[4]Recall that, as a consequence of Luzin's theorem [Rudin (1987), Chap. 2, Th. 2.24], a function $f \in L^\infty([a,b],\mathbb{R})$ is, almost everywhere with respect to the Lebesgue measure on $[a,b]$, the pointwise limit of a sequence of compactly supported continuous functions. See also [Rudin (1987), Chap. 9, Sect. 9.22].