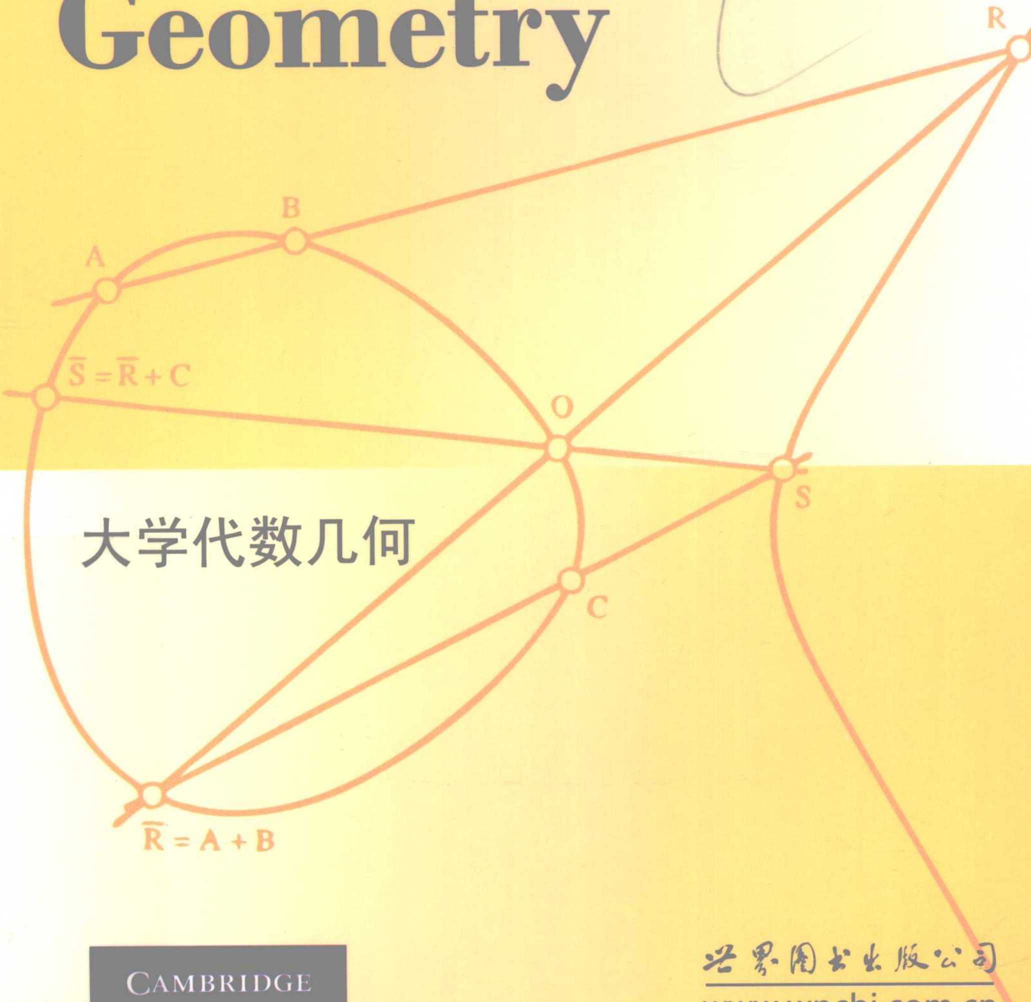
Miles Reid

Undergraduate Algebraic Geometry



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Preface

There are several good recent textbooks on algebraic geometry at the graduate level, but not (to my knowledge) any designed for an undergraduate course. Humble notes are from a course given in two successive years in the 3rd year of the Warwick undergraduate math course, and are intended as a self-contained introductory textbook.

Contents

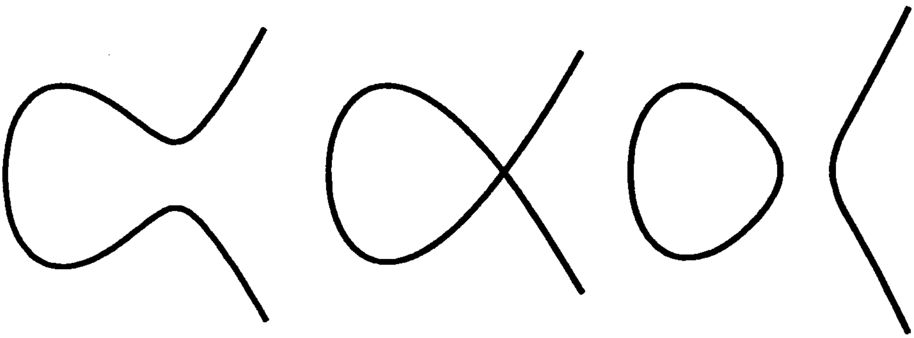
§0. Woffle

This section is intended as a cultural introduction, and is not *logically* part of the course, so just skip through it.

(0.1) A variety is (roughly) a locus defined by polynomial equations:

$$V = \{ P \in k^n \mid f_i(P) = 0 \} \subset k^n,$$

where k is a field and $f_i \in k[X_1, \ldots X_n]$ are polynomials; so for example, the plane curves $C: (f(x, y) = 0) \subset \mathbb{R}^2$ or $\mathbb{C}^2$.



$y^2 = (x + 1)(x^2 + \varepsilon)$ $\qquad$ $y^2 = (x + 1)x^2$ $\qquad$ $y^2 = (x + 1)(x^2 - \varepsilon)$

I want to study V; several questions present themselves:

Number Theory. For example, if $k = \mathbb{Q}$ and $V \subset \mathbb{Q}^n$, how can we tell if V is nonempty, or find all its points if it is? A specific case is historically of some significance: how many solutions are there to

$$x^n + y^n = 1, \quad x, y \in \mathbb{Q}, \ n \geq 3\,?$$

Questions of this kind are generally known as *Diophantine problems*.

Topology. If k is $\mathbb{R}$ or $\mathbb{C}$ (which it quite often is), what kind of topological space is V? For example, the connected components of the above cubics are obvious topological invariants.

Singularity theory. What kind of topological space is V near $P \in V$; if $f\colon V_1 \to V_2$ is a regular map between two varieties (for example, a polynomial map $\mathbb{R}^2 \to \mathbb{R}$), what kind of topology and geometry does f have near $P \in V_1$?

(0.2) There are two possible approaches to studying varieties:

Particular. Given specific polynomials f_i, we can often understand the variety V by explicit tricks with the f_i; this is fun if the dimension n and the degrees of the f_i are small, or the f_i are specially nice, but things get progressively more complicated, and there rapidly comes a time when mere ingenuity with calculations doesn't tell you much about the problem.

General. The study of properties of V leads at once to basic notions such as regular functions on V, nonsingularity and tangent planes, the dimension of a variety: the idea that curves such as the above cubics are 1-dimensional is familiar from elementary Cartesian geometry, and the pictures suggest at once what singularity should mean.

Now a basic problem in giving an undergraduate algebraic geometry course is that an adequate treatment of the 'general' approach involves so many definitions that they fill the entire course and squeeze out all substance. Therefore one has to compromise, and my solution is to cover a small subset of the general theory, with constant reference to specific examples. These notes therefore contain only a fraction of the 'standard bookwork' which would form the compulsory core of a 3-year undergraduate math course devoted entirely to algebraic geometry. On the other hand, I hope that each section contains some exercises and worked examples of substance.

(0.3) The specific flavour of algebraic geometry comes from the use of only polynomial functions (together with rational functions); to explain this, if $U \subset \mathbb{R}^2$ is an open interval, one can reasonably consider the following rings of functions on U:

$C^0(U)$ = all continuous functions $f\colon U \to \mathbb{R}$;

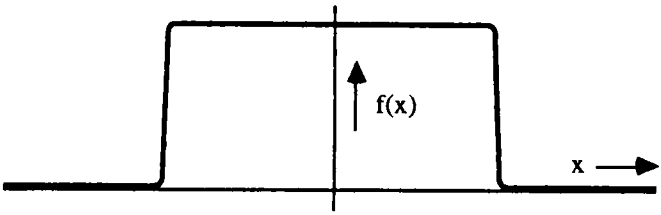
$C^\infty(U)$ = all smooth functions (that is, differentiable to any order);

$C^\omega(U)$ = all analytic functions (that is, convergent power series);

$\mathbb{R}[X]$ = the polynomial ring, viewed as polynomial functions on U.

There are of course inclusions $\mathbb{R}[X] \subset C^\omega(U) \subset C^\infty(U) \subset C^0(U)$.

These rings of functions correspond to some of the important categories of geometry: $C^0(U)$ to the topological category, $C^\infty(U)$ to the differentiable category (differentiable manifolds), C^ω to real analytic geometry, and $\mathbb{R}[X]$ to algebraic geometry. The point I want to make here is that each of these inclusion signs represents an absolutely *huge* gap, and that this leads to the main characteristics of geometry in the different categories. Although it's not stressed very much in school and first year university calculus, any reasonable way of measuring $C^0(U)$ will reveal that the differentiable functions have measure 0 in the continuous functions (so if you pick a continuous function at random then with probability 1 it will be nowhere differentiable, like Brownian motion). The gap between $C^\omega(U)$ and $C^\infty(U)$ is exemplified by the behaviour of $\exp(-1/x^2)$, the standard function which is differentiable infinitely often, but for which the Taylor series (at 0) does not converge to f; using this, you can easily build a C^∞ 'bump function' $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $|x| \le 0.9$, and $f(x) = 0$ if $|x| \ge 1$:



a C^∞ bump function

In contrast, an analytic function on U extends (as a convergent power series) to an analytic function of a complex variable on a suitable domain in $\mathbb{C}$, so that (using results from complex analysis), if $f \in C^\omega(U)$ vanishes on a real interval, it must vanish identically. This is a kind of 'rigidity' property which characterises analytic geometry as opposed to differential topology.

(0.4) There are very few polynomial functions: the polynomial ring $\mathbb{R}[X]$ is just a countable-dimensional $\mathbb{R}$-vector space, whereas $C^\omega(U)$ is already uncountable. Even allowing rational functions - that is, extending $\mathbb{R}[X]$ to its field of fractions $\mathbb{R}(X)$ - doesn't help much. (2.2) will provide an example of the characteristic rigidity of the algebraic category. The fact that it is possible to construct a geometry using only this set of functions is itself quite remarkable. Not surprisingly, there are difficulties involved in setting up this theory:

Foundations via commutative algebra. Topology and differential topology can rely on the whole corpus of ε-δ analysis taught in a series of 1st and 2nd year undergraduate courses; to do algebraic geometry working only with polynomial rings, we need instead to study rings such as the polynomial ring $k[X_1,.. X_n]$ and their ideals. In other words, we have to develop commutative algebra in place of calculus. The Nullstellensatz (§3 below) is a typical example of a statement having direct intuitive geometric content (essentially, 'different ideals of functions in $k[X_1,.. X_n]$ define different varieties $V \subset k^n$') whose proof involves quite a lengthy digression through finiteness conditions in commutative algebra.

Rational maps and functions. Another difficulty arising from the decision to work with polynomials is the necessity of introducing 'partially-defined functions'; because of the 'rigidity' hinted at above, we'll see that for some varieties (in fact for all projective varieties), there do not exist any nonconstant regular functions (see Ex. 5.1, Ex. 5.12 and the discussion in (8.10)). Rational functions (that is, 'functions' of the form $f = g/h$, where g, h are polynomial functions) are not defined at points where the denominator vanishes. Although reprehensible, it is a firmly entrenched tradition among algebraic geometers to use 'rational function' and 'rational map' to mean 'only partially-defined function (or map)'. So a rational map $f\colon V_1 \dashrightarrow V_2$ is not a map at all; the broken arrow here is also becoming traditional. Students who disapprove are recommended to give up at once and take a reading course in Category Theory instead.

This is not at all a frivolous difficulty. Even regular maps (= morphisms, these are genuine maps) have to be defined as rational maps which are regular at all points $P \in V$ (that is, well defined, the denominator can be chosen not to vanish at P). Closely related to this is the difficulty of giving a proper intrinsic definition of a variety: in this course (and in others like it, in my experience), affine varieties $V \subset \mathbb{A}^n$ and quasiprojective varieties $V \subset \mathbb{P}^n$ will be defined, but there will be no proper definition of 'variety' without reference to an ambient space. Roughly speaking, a variety should be what you get if you glue together a number of affine varieties along isomorphic open subsets. But our present language, in which isomorphisms are themselves defined more or less explicitly in terms of rational functions, is just too cumbersome; the proper language for this glueing is sheaves, which are well treated in graduate textbooks.

(0.5) So much for the drawbacks of the algebraic approach to geometry. Having said this, almost all the algebraic varieties of importance in the world today are quasiprojective, and we can have quite a lot of fun with varieties without worrying overmuch about the finer points of definition.

The main advantages of algebraic geometry are that it is purely algebraically defined, and that it applies to any field, not just $\mathbb{R}$ or $\mathbb{C}$; we can do geometry over fields of characteristic p. Don't say 'characteristic p – big deal, that's just the finite fields'; to start with, very substantial parts of group theory are based on geometry over finite fields, as are large parts of combinatorics used in computer science. Next, there are lots of interesting fields of characteristic p other than finite ones. Moreover, at a deep level, the finite fields are present and working inside $\mathbb{Q}$ and $\mathbb{C}$. Most of the deep results on arithmetic of varieties over $\mathbb{Q}$ use a considerable amount of geometry over $\mathbb{C}$ or over the finite fields and their algebraic closures.

This concludes the introduction; see the informal discussion in (2.15) and the final §8 for more general culture.

(0.6) As to the structure of the book, Chapter I and Chapter III aim to indicate some worthwhile problems which can be studied by means of algebraic geometry. Chapter II is an introduction to the commutative algebra referred to in (0.4) and to the categorical framework of algebraic geometry; the student who is prone to headaches could perhaps take some of the proofs for granted here, since the material is standard, and the author is a professional algebraic geometer of the highest moral fibre.

§8 contains odds and ends that may be of interest or of use to the student, but that don't fit in the main text: a little of the history and sociology of the modern subject, hints as to relations of the subject-matter with more advanced topics, technical footnotes, etc.

Prerequisites for this course:

Algebra: Quadratic forms, easy properties of commutative rings and their ideals, principal ideal domains and unique factorisation.

Galois Theory: Fields, polynomial rings, finite extensions, algebraic versus transcendental extensions, separability.

Topology and geometry: Definition of topological space, projective space $\mathbb{P}^n$ (but I'll go through it again in detail).

Calculus in $\mathbb{R}^n$: Partial derivatives, implicit function theorem (but I'll remind you of what I need when we get there).

Commutative algebra: Other experience with commutative rings is desirable, but not essential.

Course relates to:

Complex Function Theory. An algebraic curve over $\mathbb{C}$ is a 1-dimensional complex manifold, and regular functions on it are holomorphic, so that this course is closely related to complex function theory, even if the relation is not immediately apparent.

Algebraic Number Theory. For example the relation with Fermat's Last Theorem.

Catastrophe Theory. Catastrophes are singularities, and are essentially always given by polynomial functions, so that the analysis of the geometry of the singularities is pure algebraic geometry.

Commutative Algebra. Algebraic geometry provides motivation for commutative algebra, and commutative algebra provides technical support for algebraic geometry, so that the two subjects enrich one another.

Exercises to §0.

0.1. (a) Show that for fixed values of (y, z), x is a repeated root of $x^3 + xy + z = 0$ if and only if $x = -3z/2y$ and $4y^3 + 27z^2 = 0$;

(b) there are 3 distinct roots if and only if $4y^3 + 27z^2 < 0$;

(c) sketch the surface $S: (x^3 + xy + z = 0) \subset \mathbb{R}^3$ and its projection onto the (y, z)-plane;

(d) now open up any book or article on catastrophe theory and compare.

0.2. Let $f \in \mathbb{R}[X, Y]$ and let $C: (f = 0) \subset \mathbb{R}^2$; say that $P \in C$ is *isolated* if there is an $\varepsilon > 0$ such that $C \cap B(P, \varepsilon) = P$. Show by example that C can have isolated points. Prove that if $P \in C$ is an isolated point then $f: \mathbb{R}^2 \to \mathbb{R}$ must have a max or min at P, and deduce that $\partial f/\partial x$ and $\partial f/\partial y$ vanish at P. This proves that an isolated point of a real curve is singular.

0.3. **Cubic curves:** (i) Draw the graph of $y = 4x^3 + 6x^2$ and its intersection with the horizontal lines $y = t$ for integer values of $t \in [-1, 3]$; (ii) draw the cubic curves $y^2 = 4x^3 + 6x^2 - t$ for the same values of t.

Books

Most of the following are textbooks at a graduate level, and some are referred to in the text:

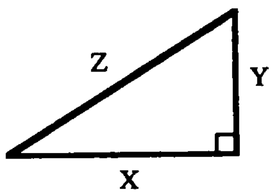
W. Fulton, Algebraic curves, Springer. (This is the most down-to-earth and self-contained of the graduate texts; Ch. 1–6 are quite well suited to an undergraduate course, although the material is somewhat dry.)
I.R. Shafarevich, Basic algebraic geometry, Springer. (A graduate text, but Ch. I, and §II.1 are quite suitable material.)
P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley. (Gives the complex analytic point of view.)
D. Mumford, Algebraic geometry I, Complex projective varieties, Springer.
D. Mumford, Introduction to algebraic geometry, Harvard notes. (Not immediately very readable, but goes directly to the main points; many algebraic geometers of my generation learned their trade from these notes. Recently reissued as Springer LNM 1358, and therefore no longer a little red book.)
K. Kendig, Elementary algebraic geometry, Springer. (Treats the relation between algebraic geometry and complex analytic geometry.)
R. Hartshorne, Algebraic geometry, Springer. (This is the professional's handbook, and covers much more advanced material; Ch. I is an undergraduate course in bare outline.)
M. Berger, Geometry I and II, Springer. (Some of the material of the sections on quadratic forms and quadric hypersurfaces in II is especially relevant.)
M.F. Atiyah and I.G. Macdonald, Commutative algebra, Addison-Wesley. (An invaluable textbook.)
E. Kunz, Introduction to commutative algebra and algebraic geometry, Birkhäuser.
H. Matsumura, Commutative ring theory, Cambridge. (A more detailed text on commutative algebra.)
D. Mumford, Curves and their Jacobians, Univ. of Michigan Press. (Colloquial lectures, going quite deep quite fast.)
C.H. Clemens, A scrapbook of complex curves, Plenum. (Lots of fun.).
E. Brieskorn and H. Knörrer, Plane algebraic curves, Birkhäuser.
A. Beauville, Complex algebraic surfaces, LMS Lecture Notes, Cambridge.
J. Kollár, The structure of algebraic threefolds: An introduction to Mori's program, Bull. Amer. Math. Soc. 17 (1987), 211–273. (A nicely presented travel brochure to one active area of research. Mostly harmless.)
J.G. Semple and L. Roth, Introduction to algebraic geometry, Oxford. (A marvellous old book, full of information, but almost entirely lacking in rigour.)
J.L. Coolidge, Treatise on algebraic plane curves, Oxford and Dover.

Chapter I. Playing with plane curves

§1. Plane conics

I start by studying the geometry of conics as motivation for the projective plane $\mathbb{P}^2$. Projective geometry is usually mentioned in 2nd year undergraduate geometry courses, and I recall some of the salient features, with some emphasis on homogeneous coordinates, although I completely ignore the geometry of linear subspaces and the 'cross-ratio'. The most important aim for the student should be to grasp the way in which geometric ideas (for example, the idea that 'points at infinity' correspond to asymptotic directions of curves) are expressed in terms of coordinates. The interplay between the intuitive geometric picture (which tells you what you should be expecting), and the precise formulation in terms of coordinates (which allows you to cash in on your intuition) is a fascinating aspect of algebraic geometry.

(1.1) Example of a parametrised curve. Pythagoras' Theorem says that, in the diagram



$$, \quad X^2 + Y^2 = Z^2,$$

so (3, 4, 5) and (5, 12, 13), as every ancient Egyptian knew. How do you find all integer solutions? The equation is homogeneous, so that $x = X/Z$, $y = Y/Z$ gives the circle $C : (x^2 + y^2 = 1) \subset \mathbb{R}^2$, which can easily be seen to be parametrised as

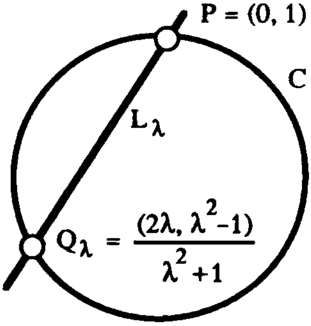
$$x = 2\lambda/(\lambda^2 + 1), \; y = (\lambda^2 - 1)/(\lambda^2 + 1), \quad \text{where} \quad \lambda = x/(1 - y);$$

so this gives all solutions:

$$X = 2\ell m, \; Y = \ell^2 - m^2, \; Z = \ell^2 + m^2 \quad \text{with} \quad \ell, m \in \mathbb{Z} \text{ coprime},$$

(or each divided by 2 if ℓ, m are both odd). Note that the equation is homogeneous, so that if (X, Y, Z) is a solution, then so is $(\lambda X, \lambda Y, \lambda Z)$.

Maybe the parametrisation was already familiar from school geometry, and in any case, it's easy to verify that it works. However, if I didn't know it already, I could have obtained it by an easy geometric argument, namely linear projection from a given point:



$P = (0, 1) \in C$, and if $\lambda \in \mathbb{Q}$ is any value, then the line L_λ through P with slope $-\lambda$ meets C in a further point Q_λ. This construction of a map by means of linear projection will appear many times in what follows.

(1.2) Similar example. $C: (2X^2 + Y^2 = 5Z^2)$. The same method leads to the parametrisation $\mathbb{R} \to C$ given by

$$x = \frac{2\sqrt{5}\lambda}{1+2\lambda^2}, \quad y = \frac{2\lambda^2 - 1}{1+2\lambda^2}.$$

This allows us to understand all about points of C with coefficients in $\mathbb{R}$, and there's no real difference from the previous example; what about $\mathbb{Q}$?

Proposition. If $(a, b, c) \in \mathbb{Q}$ satisfies $2a^2 + b^2 = 5c^2$ then $(a, b, c) = (0, 0, 0)$.

Proof. Multiplying through by a common denominator and taking out a common factor if necessary, I can assume that a, b, c are integers, not all of which are divisible by 5; also if $5 \mid a$ and $5 \mid b$ then $25 \mid 5c^2$, so that $5 \mid c$, which contradicts what I've just said. It is now easy to get a contradiction by considering the possible values of a and b mod 5: since any square is 0, 1 or 4 mod 5, clearly $2a^2 + b^2$ is one of 0+1, 0+4, 2+0, 2+1, 2+4, 8+0, 8+1 or 8+4 mod 5, none of which can be of the form $5c^2$. Q.E.D.

Note that this is a thoroughly arithmetic argument.