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Enumerative Combinatorics

Volume 2

RICHARD P. STANLEY

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麻省理工学院

机械工业出版社
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Foreword

Most textbooks written in our day have a short half-life. Published to meet the demands of a lucrative but volatile market, inspired by the table of contents of some out-of-print classic, garnished with multicolored tables, enhanced by nutshell summaries, enriched by exercises of dubious applicability, they decorate the shelves of college bookstores come September. The leftovers after Registration Day will be shredded by Christmas, unwanted even by remainder bookstores. The pageant is repeated every year, with new textbooks on the same shelves by other authors (or a new edition if the author is the same), as similar to the preceding as one can make them, short of running into copyright problems.

Every once in a long while, a textbook worthy of the name comes along; invariably, it is likely to prove *aere perennius*: Weber, Bertini, van der Waerden, Feller, Dunford and Schwartz, Ahlfors, Stanley.

The mathematical community professes a snobbish distaste for expository writing, but the facts are at variance with the words. In actual reality, the names of authors of the handful of successful textbooks written in this century are included in the list of the most celebrated mathematicians of our time.

Only another textbook writer knows the pains and the endless effort that goes into this kind of writing. The amount of time that goes into drafting a satisfactory exposition is always underestimated by the reader. The time required to complete one single chapter exceeds the time required to publish a research paper. But far from wasting his or her time, the author of a successful textbook will be amply rewarded by a renown that will spill into the distant future. History is more likely to remember the name of the author of a definitive exposition than the names of many a research mathematician.

I find it impossible to predict when Richard Stanley's two-volume exposition of combinatorics may be superseded. No one will dare try, let alone be able, to match the thoroughness of coverage, the care for detail, the definitiveness of proof, the elegance of presentation. Stanley's book possesses that rarest quality among textbooks: you can open it at any page and start reading with interest without having to hark back to page one for previous explanations.

Combinatorics, which only thirty years ago was a fledgling among giants, may well be turning out to be a greater giant, thanks largely to Richard Stanley's work. Every one who deals with discrete mathematics, from category theorists to molecular biologists, owes him a large debt of gratitude.

Gian-Carlo Rota
March 21, 1998

Preface

This is the second (and final) volume of a graduate-level introduction to enumerative combinatorics. To those who have been waiting twelve years since the publication of Volume 1, I can only say that no one is more pleased to see Volume 2 finally completed than myself. I have tried to cover what I feel are the fundamental topics in enumerative combinatorics, and the ones that are the most useful in applications outside of combinatorics. Though the book is primarily intended to be a textbook for graduate students and a resource for professional mathematicians, I hope that undergraduates and even bright high-school students will find something of interest. For instance, many of the 66 combinatorial interpretations of Catalan numbers provided by Exercise 6.19 should be accessible to undergraduates with a little knowledge of combinatorics.

Much of the material in this book has never appeared before in textbook form. This is especially true of the treatment of symmetric functions in Chapter 7. Although the theory of symmetric functions and its connections with combinatorics is in my opinion one of the most beautiful topics in all of mathematics, it is a difficult subject for beginners to learn. The superb book by Macdonald on symmetric functions is highly algebraic and eschews the fundamental combinatorial tool in this subject, viz., the Robinson–Schensted–Knuth algorithm. I hope that Chapter 7 adequately fills this gap in the mathematical literature. Chapter 7 should be regarded as only an introduction to the theory of symmetric functions, and not as a comprehensive treatment.

As in Volume 1, the exercises play a vital role in developing the subject. If in reading the text the reader is left with the feeling of "what's it good for?" and is not satisfied with the applications presented there, then (s)he should turn to the exercises. Thanks to the wonders of electronic word processing, I found it much easier than for Volume 1 to assemble a wide variety of exercises and solutions.

I am grateful to the many persons who have contributed in a number of ways to the improvement of this book. Special thanks go to Sergey Fomin for his many suggestions related to Chapter 7, and especially for his masterful exposition of the difficult material of Appendix 1. Other persons who have carefully read portions of earlier versions of the book and who have offered valuable suggestions are Christine Bessenrodt, Francesco Brenti, Persi Diaconis, Wungkum Fong, Phil Hanlon, and Michelle Wachs. Robert Becker typed most of Chapter 5, and Tom Roby and Bonnie Friedman provided invaluable TeX assistance. The following persons at Cambridge University Press and TechBooks have been a pleasure to work with throughout the writing and production of this book: Catherine Felgar, Shamus McGillicuddy, Andrew Wilson, and especially Lauren Cowles, whose patience and support is greatly appreciated. The following additional persons have made at least one significant contribution that is not explicitly mentioned in the text, and I regret if I have inadvertently omitted anyone else who belongs on this list: Christos Athanasiadis, Anders Björner, Mireille Bousquet-Mélou, Bradley Brock, David Buchsbaum, Emeric Deutsch, Kimmo Eriksson, Dominique Foata, Ira Gessel, Curtis Greene, Patricia Hersh, Martin Isaacs, Benjamin Joseph, Martin Klazar, Donald Knuth, Darla Kremer, Valery Liskovets, Peter Littelmann, Ian Macdonald, Alexander Mednykh, Thomas Müller, Andrew Odlyzko, Alexander Postnikov, Robert Proctor, Douglas Rogers, Lou Shapiro, Rodica Simion, Mark Skandera, Louis Solomon, Dennis Stanton, Robert Sulanke, Sheila Sundaram, Jean-Yves Thibon, and Andrei Zelevinsky.

Richard Stanley
Cambridge, Massachusetts
March 1998

The paperback printing contains addenda to some of the exercises in a new section on page 583. The exercises in question are indicated by * in the main text.

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p. 124, Exercise 5.28
p. 136, Exercise 5.41(j)
p. 144, Exercise 5.53
p. 151, Exercise 5.62(b)
p. 231, Exercise 6.25(i)
p. 232, Exercise 6.27(c)
p. 264, Exercise 6.19(iii)
p. 265, Exercise 6.19(mmm)
p. 272, Exercise 6.33(c)
p. 279, Exercise 6.56(c)
p. 467, Exercise 7.55(b)
p. 534, Exercise 7.74
p. 539, Exercise 7.85

Notation

The notation follows that of Volume 1, with the following exceptions.

- The coefficient of x^n in the power series $F(x)$ is now denoted $[x^n]F(x)$. This notation is generalized in an obvious way to such situations as

$$[x^m y^n] \sum_{i,j} a_{ij} x^i y^j = a_{mn}$$

$$\left[\frac{x^n}{n!}\right] \sum_i a_i \frac{x^i}{i!} = a_n.$$

- The number of inversions, number of descents, and major index of a permutation (or more generally of a sequence) w are denoted $\mathrm{inv}(w)$, $\mathrm{des}(w)$, and $\mathrm{maj}(w)$, respectively, rather than $i(w)$, $d(w)$, and $\iota(w)$. Sometimes, especially when we are regarding the symmetric group $\mathfrak{S}_n$ as a Coxeter group, we write $\ell(w)$ instead of $\mathrm{inv}(w)$.

The following notation is used for various rings and fields of generating functions. Here K denotes a field, which is always the field of coefficients of the series below. All Laurent series and fractional Laurent series are understood to have only finitely many terms with negative exponents.

$K[x]$	ring of polynomials in x
$K(x)$	field of rational functions in x (the quotient field of $K[x]$)
$K[[x]]$	ring of formal (power) series in x
$K((x))$	field of Laurent series in x (the quotient field of $K[[x]]$)
$K_{\mathrm{alg}}[[x]]$	ring of algebraic power series in x over $K(x)$
$K_{\mathrm{alg}}((x))$	field of algebraic Laurent series in x over $K(x)$
$K^{\mathrm{fra}}[[x]]$	ring of fractional power series in x

$K^{\mathrm{fra}}((x))$	field of fractional Laurent series in x (the quotient field of $K^{\mathrm{fra}}[[x]]$)
$K\langle X\rangle$	ring of noncommutative polynomials in the alphabet (set of variables) X
$K_{\mathrm{rat}}\langle\langle X\rangle\rangle$	ring of rational (= recognizable) noncommutative series in the alphabet X
$K\langle\langle X\rangle\rangle$	ring of formal (noncommutative) series in the alphabet X
$K_{\mathrm{alg}}\langle\langle X\rangle\rangle$	ring of (noncommutative) algebraic series in the alphabet X

Enumerative Combinatorics

This is the second of a two-volume basic introduction to enumerative combinatorics at a level suitable for graduate students and research mathematicians.

This volume covers the composition of generating functions, trees, algebraic generating functions, D-finite generating functions, noncommutative generating functions, and symmetric functions. The chapter on symmetric functions provides the only available treatment of this subject suitable for an introductory graduate course and focusing on combinatorics, especially the Robinson–Schensted–Knuth algorithm. Also covered are connections between symmetric functions and representation theory. An appendix (written by Sergey Fomin) covers some deeper aspects of symmetric function theory, including jeu de taquin and the Littlewood–Richardson rule.

As in Volume 1, the exercises play a vital role in developing the material. There are over 250 exercises, all with solutions or references to solutions, many of which concern previously unpublished results.

Graduate students and research mathematicians who wish to apply combinatorics to their work will find this an authoritative reference.

Richard P. Stanley is Professor of Applied Mathematics at the Massachusetts Institute of Technology. He has held visiting positions at UCSD, the University of Strasbourg, California Institute of Technology, the University of Augsburg, Tokai University, and the Royal Institute of Technology in Stockholm. He has published over 100 research papers in algebraic combinatorics. In addition to the two-volume *Enumerative Combinatorics*, he has published one other book, *Combinatorics and Commutative Algebra* (Birkhäuser; second edition, 1997). He is a fellow of the American Academy of Arts and Sciences, a member of the National Academy of Sciences, and a recipient of the Pólya Prize in Applied Combinatorics awarded by the Society for Industrial and Applied Mathematics.

Contents

5

Trees and the Composition of Generating Functions

5.1 The Exponential Formula

If $F(x)$ and $G(x)$ are formal power series with $G(0)=0$, then we have seen (after Proposition 1.1.9) that the composition $F(G(x))$ is a well-defined formal power series. In this chapter we will investigate the combinatorial ramifications of power series composition. In this section we will be concerned with the case where $F(x)$ and $G(x)$ are exponential generating functions, and especially the case $F(x) = e^x$.

Let us first consider the combinatorial significance of the product $F(x)G(x)$ of two exponential generating functions

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}$$

$$G(x) = \sum_{n\geq 0} g(n)\frac{x^n}{n!}.$$

Throughout this chapter K denotes a field of characteristic 0 (such as $\mathbb{C}$ with some indeterminates adjoined). We also denote by $E_f(x)$ the exponential generating function of the function $f : \mathbb{N} \to K$, i.e.,

$$E_f(x) = \sum_{n\geq 0} f(n)\frac{x^n}{n!}.$$

5.1.1 Proposition. *Given functions $f, g : \mathbb{N} \to K$, define a new function $h : \mathbb{N} \to K$ by the rule*

$$h(\#X) = \sum_{(S,T)} f(\#S)g(\#T), \tag{5.1}$$

where X is a finite set, and where (S,T) ranges over all weak ordered partitions of X into two blocks, i.e., $S \cap T = \emptyset$ and $S \cup T = X$. Then

$$E_h(x) = E_f(x)E_g(x). \tag{5.2}$$

Proof. Let $\#X = n$. There are $\binom{n}{k}$ pairs (S, T) with $\#S = k$ and $\#T = n - k$, so

$$h(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) g(n-k).$$

From this (5.2) follows. □

One could also prove Proposition 5.1.1 by using Theorem 3.15.4 applied to the binomial poset $\mathbb{B}$ of Example 3.15.3.

We have stated Proposition 5.1.1 in terms of a certain relationship (5.1) among functions f, g, and h, but it is important to understand its combinatorial significance. Suppose we have two types of structures, say α and β, which can be put on a finite set X. We assume that the allowed structures depend only on the cardinality of X. A new "combined" type of structure, denoted $\alpha \cup \beta$, can be put on X by placing structures of type α and β on subsets S and T, respectively, of X such that $S \cup T = X$, $S \cap T = \emptyset$. If $f(k)$ (respectively $g(k)$) are the number of possible structures on a k-set of type α (respectively, β), then the right-hand side of (5.1) counts the number of structures of type $\alpha \cup \beta$ on X. More generally, we can assign a weight $w(\Gamma)$ to any structure Γ of type α or β. A combined structure of type $\alpha \cup \beta$ is defined to have weight equal to the product of the weights of each part. If $f(k)$ and $g(k)$ denote the sums of the weights of all structures on a k-set of types α and β, respectively, then the right-hand side of (5.1) counts the sum of the weights of all structures of type $\alpha \cup \beta$ on X.

5.1.2 Example. Given an n-element set X, let $h(n)$ be the number of ways to split X into two subsets S and T with $S \cup T = X$, $S \cap T = \emptyset$, and then to linearly order the elements of S and to choose a subset of T. There are $f(k) = k!$ ways to linearly order a k-element set, and $g(k) = 2^k$ ways to choose a subset of a k-element set. Hence

$$\begin{aligned} \sum_{n\geq 0} h(n)\frac{x^n}{n!} &= \left(\sum_{n\geq 0} n!\frac{x^n}{n!}\right)\left(\sum_{n\geq 0} 2^n\frac{x^n}{n!}\right) \\ &= \frac{e^{2x}}{1-x}. \end{aligned}$$

Proposition 5.1.1 can be iterated to yield the following result.

5.1.3 Proposition. *Fix $k \in \mathbb{P}$ and functions $f_1, f_2, \ldots, f_k : \mathbb{N} \to K$. Define a new function $h : \mathbb{N} \to K$ by*

$$h(\#S) = \sum f_1(\#T_1) f_2(\#T_2) \cdots f_k(\#T_k),$$

where $(T_1, \ldots, T_k)$ ranges over all weak ordered partitions of S into k blocks, i.e., $T_1, \ldots, T_k$ are subsets of S satisfying: (i) $T_i \cap T_j = \emptyset$ if $i \neq j$, and (ii) $T_1 \cup \cdots \cup T_k = S$. Then

$$E_h(x) = \prod_{i=1}^{k} E_{f_i}(x).$$

We are now able to give the main result of this section, which explains the combinatorial significance of the composition of exponential generating functions.

5.1.4 Theorem (The Compositional Formula). *Given functions $f : \mathbb{P} \to K$ and $g : \mathbb{N} \to K$ with $g(0) = 1$, define a new function $h : \mathbb{N} \to K$ by*

$$h(\#S) = \sum_{\pi=\{B_1,\ldots,B_k\}\in\Pi(S)} f(\#B_1)f(\#B_2)\cdots f(\#B_k)g(k), \quad \#S > 0, \qquad (5.3)$$

$$h(0) = 1,$$

where the sum ranges over all partitions (as defined in Section 1.4) $\pi = \{B_1, \ldots, B_k\}$ of the finite set S. Then

$$E_h(x) = E_g(E_f(x)).$$

(Here $E_f(x) = \sum_{n\geq 1} f(n)x^n/n!$, since f is only defined on positive integers.)

Proof. Suppose $\#S = n$, and let $h_k(n)$ denote the right-hand side of (5.3) for *fixed* k. Since $B_1, \ldots, B_k$ are nonempty, they are all distinct, so there are $k!$ ways of linearly ordering them. Thus by Proposition 5.1.3,

$$E_{h_k}(x) = \frac{g(k)}{k!} E_f(x)^k. \qquad (5.4)$$

Summing (5.4) over all $k \geq 1$ yields the desired result. □

Theorem 5.1.4 has the following combinatorial significance. Many structures on a set, such as graphs or posets, may be regarded as disjoint unions of their connected components. In addition, some additional structure may be placed on the components themselves, e.g., the components could be linearly ordered. If there are $f(j)$ connected structures on a j-set and $g(k)$ ways to place an additional structure on k components, then $h(n)$ is the total number of structures on an n-set. There is an obvious generalization to weighted structures, such as was discussed after Proposition 5.1.1.

The following example should help to elucidate the combinatorial meaning of Theorem 5.1.4; more substantial applications are given in Section 5.2.

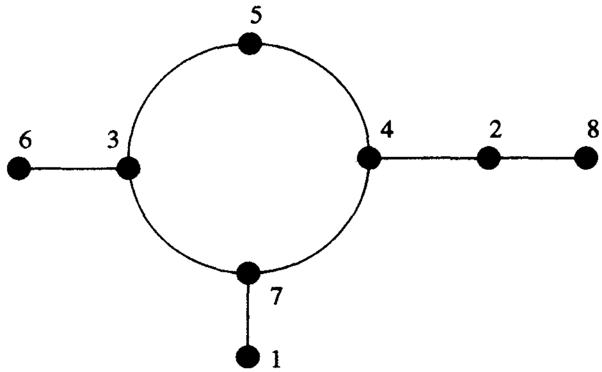

Figure 5-1. A circular arrangement of lines.

5.1.5 Example. Let $h(n)$ be the number of ways for n persons to form into nonempty lines, and then to arrange these lines in a circular order. Figure 5-1 shows one such arrangement of nine persons. There are $f(j) = j!$ ways to linearly order j persons, and $g(k) = (k-1)!$ ways to circularly order $k \geq 1$ lines. Thus

$$E_f(x) = \sum_{n\geq 1} n!\frac{x^n}{n!} = \frac{x}{1-x},$$

$$E_g(x) = 1 + \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = 1 + \log(1-x)^{-1},$$

so

$$\begin{aligned} E_h(x) &= E_g(E_f(x)) \\ &= 1 + \log\left(1 - \frac{x}{1-x}\right)^{-1} \\ &= 1 + \log(1-2x)^{-1} - \log(1-x)^{-1} \\ &= 1 + \sum_{n\geq 1}(2^n - 1)(n-1)!\frac{x^n}{n!}, \end{aligned}$$

whence $h(n) = (2^n - 1)(n-1)!$. Naturally, such a simple answer demands a simple combinatorial proof. Namely, arrange the n persons in a circle in $(n-1)!$ ways. In each of the n spaces between two persons, either do or do not draw a bar, except that at least one bar must be drawn. There are thus $2^n - 1$ choices for the bars. Between two consecutive bars (or a bar and itself if there is only one bar) read

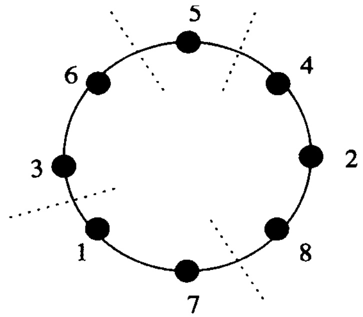

Figure 5-2. An equivalent form of Figure 5-1.

the persons in clockwise order to obtain their order in line. See Figure 5-2 for this method of representing Figure 5-1.

The most common use of Theorem 5.1.4 is the case where $g(k) = 1$ for all k. In combinatorial terms, a structure is put together from "connected" components, but no additional structure is placed on the components themselves.

5.1.6 Corollary (The Exponential Formula). *Given a function $f : \mathbb{P} \to K$, define a new function $h : \mathbb{N} \to K$ by*

$$h(\#S) = \sum_{\pi=\{B_1,\ldots,B_k\}\in\Pi(S)} f(\#B_1)f(\#B_2)\cdots f(\#B_k), \qquad \#S > 0, \tag{5.5}$$
$$h(0) = 1.$$

Then

$$E_h(x) = \exp E_f(x). \tag{5.6}$$

Let us say a brief word about the computational aspects of equation (5.6). If the function $f(n)$ is given, then one can use (5.5) to compute $h(n)$. However, there is a much more efficient way to compute $h(n)$ from $f(n)$ (and conversely).

5.1.7 Proposition. *Let $f : \mathbb{P} \to K$ and $h : \mathbb{N} \to K$ be related by $E_h(x) = \exp E_f(x)$ (so in particular $h(0) = 1$). Then we have for $n \geq 0$ the recurrences*

$$h(n+1) = \sum_{k=0}^{n} \binom{n}{k} h(k) f(n+1-k), \tag{5.7}$$

$$f(n+1) = h(n+1) - \sum_{k=1}^{n} \binom{n}{k} h(k) f(n+1-k). \tag{5.8}$$