Dietrich Braess

Finite elements

Theory, fast solvers, and applications in solid mechanics

Second Edition

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Preface to the Second English Edition

The theory of finite elements and their applications is so vivid that the second printing gave rise to some additions. We have not only eliminated some misprints, but we have also added some new material that although being basic has turned out to be of interest in actual research or actual applications of finite elements during the last years. We will emphasize some of these extensions.

The introduction of finite element spaces in Chapter II, §5 is now focused such that all the ingredients of the formal definition at the end of that § are well motivated.

The general considerations of saddle point problems in Chapter III are augmented. The direct and converse theorems that are related to Fortin interpolation are presented now under a common aspect. Mixed methods are often connected with a softening of the energy functional that is wanted in some applications for good reasons. It is described in order to understand a different but equivalent variational formulation that has become popular in solid mechanics.

In Chapter IV only the standard proof of the Kantorowitch inequality has been replaced by a shorter one.

The multigrid theory requires less regularity assumptions if convergence with respect to the energy norm is considered. A quick introduction into that theory is now included, and multigrid algorithms are also considered in the framework of space decompositions.

Finite element computations in solid mechanics require often appropriate elements in order to avoid an effect called "locking" by engineers. From the mathematical point of view we have problems with a small parameter. Methods for treating nearly incompressible material serve as a model for positive results while negative results are easily described for a more general framework.

The author wants to thank numerous friends who have given valuable hints for improvements of the text. Finally thanks are going to Cambridge University Press for the continuation of the good cooperation.

Autumn, 2000 *Dietrich Braess*

Preface to the First English Edition

This book is based on lectures regularly presented to students in the third and fourth year at the Ruhr-University, Bochum. It was also used by the translater, Larry Schumaker, in a graduate course at Vanderbilt University in Nashville. I would like to thank him for agreeing to undertake the translation, and for the close cooperation in carrying it out. My thanks are also due to Larry and his students for raising a number of questions which led to improvements in the material itself.

Chapters I and II and selected sections of Chapters III and V provide material for a typical course. I have especially emphasized the differences with the numerical treatment of ordinary differential equations (for more details, see the preface to the German edition).

One may ask why I was not content with presenting only simple finite elements based on complete polynomials. My motivation for doing more was provided by problems in fluid mechanics and solid mechanics, which are treated to some extent in Chapter III and VI. I am not aware of other textbooks for mathematicians which give a mathematical treatment of finite elements in solid mechanics in this generality.

The English translation contains some additions as compared to the German edition from 1992. For example, I have added the theory for basic a posteriori error estimates since a posteriori estimates are often used in connection with local mesh refinements. This required a more general interpolation process which also applies to non-uniform grids. In addition, I have also included an analysis of locking phenomena in solid mechanics.

Finally, I would like to thank Cambridge University Press for their friendly cooperation, and also Springer-Verlag for agreeing to the publication of this English version.

Autumn, 1996 *Dietrich Braess*

Preface to the German Edition

The method of finite elements is one of the main tools for the numerical treatment of elliptic and parabolic partial differential equations. Because it is based on the variational formulation of the differential equation, it is much more flexible than finite difference methods and finite volume methods, and can thus be applied to more complicated problems. For a long time, the development of finite elements was carried out in parallel by both mathematicians and engineers, without either group acknowledging the other. By the end of the 60's and the beginning of the 70's, the material became sufficiently standardized to allow its presentation to students. This book is the result of a series of such lectures.

In contrast to the situation for ordinary differential equations, for elliptic partial differential equations, frequently no classical solution exists, and we often have to work with a so-called weak solution. This has consequences for both the theory and the numerical treatment. While it is true that classical solutions do exist under approriate regularity hypotheses, for numerical calculations we usually cannot set up our analisis in a framework in which the existence of classical solutions is guaranteed.

One way to get a suitable framework for solving elliptic boundary-value problems using finite elements is to pose them as variational problems. It is our goal in Chapter II to present the simplest possible introduction to this approach. In Sections 1 – 3 we discuss the existence of weak solutions in Sobolev spaces, and explain how the boundary conditions are incorporated into the variational calculation. To give the reader a feeling for the theory, we derive a number of properties of Sobolev spaces, or at least illustrate them. Sections 4 – 8 are devoted to the foundations of finite elements. The most difficult part of this chapter is §6 where approximation theorems are presented. To simplify matters, we first treat the special case of regular grids, which the reader may want to focus on in a first reading.

In Chapter III we come to the part of the theory of finite elements which requires deeper results from functional analysis. These are presented in §3. Among other things, the reader will learn about the famous Ladyshenskaja–Babuška–Brezzi condition, which is of great importance for the proper treatment of problems in fluid mechanics and for mixed methods in structural mechanics. In fact, without this knowledge and relying only on *common sense*, we would very likely find ourselves trying to solve problems in fluid mechanics using elements with an unstable behavior.

It was my aim to present this material with as little reliance on results from real analysis and functional analysis as possible. On the other hand, a certain basic knowledge is extremely useful. In Chapter I we briefly discuss the difference between the different types of partial differential equations. Students confronting the numerical solution of elliptic differential equations for the first time often find the finite difference method more accessible. However, the limits of the method usually become apparent only later. For completeness we present an elementary introduction to finite difference methods in Chapter I.

For fine discretizations, the finite element method leads to very large systems of equations. The operation count for solving them by direct methods grows like n^2. In the last two decades, very efficient solvers have been developed based on *multigrid methods and on the method of conjugate gradients*. We treat these subjects in detail in Chapters IV and V.

Structural mechanics provides a very important application area for finite elements. Since these kinds of problems usually involve systems of partial differential equations, often the elementary methods of Ch. II do not suffice, and we have to use the extra flexibility which the deeper results of Ch. III allow. I found it necessary to assemble a surprisingly wide set of building blocks in order to present a mathematically rigorous theory for the numerical treatment by finite elements of problems in linear elasticity theory.

Almost every section of the book includes a set of Problems, which are not only excercises in the strict sense, but also serve to further develop various formulae or results from a different viewpoint, or to follow a topic which would have disturbed the flow had it been included in the text itself. It is well-known that in the numerical treatment of partial differential equations, there are many opportunities to go down a false path, even if unintended, particularly if one is thinking in terms of classical solutions. Learning to avoid such pitfalls is one of the goals of this book.

This book is based on lectures regularly presented to students in the fifth through eighth semester at the Ruhr University, Bochum. Chapters I and II and parts of Chapters III and V were presented in one semester, while the method of conjugate gradients was left to another course. Chapter VI is the result of my collaboration with both mathematicians and engineers at the Ruhr University.

A text like this can only be written with the help of many others. I would like to thank F.-J. Barthold, C. Blömer, H. Blum, H. Cramer, W. Hackbusch, A. Kirmse, U. Langer, P. Peisker, E. Stein, R. Verfürth, G. Wittum and B. Worat for their corrections and suggestions for improvements. My thanks are also due to Frau L. Mischke, who typeset the text using TEX, and to Herr Schwarz for his help with technical problems relating to TEX. Finally, I would like to express my appreciation to Springer-Verlag for the publication of the German edition of this book, and for the always pleasant collaboration on its production.

Bochum, Autumn, 1991 *Dietrich Braess*

Notation

Notation for Differential Equations and Finite Elements

Ω	open set in $\mathbb{R}^n$
Γ	$=\partial\Omega$
Γ_D	part of the boundary on which Dirichlet conditions are prescribed
Γ_N	part of the boundary on which Neumann conditions are prescribed
Δ	Laplace operator
L	differential operator
a_{ik}, a_0	coefficient functions of the differential equation
$[\cdot]_*$	difference star, stencil
$L^2(\Omega)$	space of square-integrable functions over Ω
$H^m(\Omega)$	Sobolev space of L_2 functions with square-integrable derivatives up to order m
$H_0^m(\Omega)$	subspace of $H^m(\Omega)$ of functions with generalized zero bounary conditions
$C^k(\Omega)$	set of functions with continuous derivatives up to order k
$C_0^k(\Omega)$	subspace of $C^k(\Omega)$ of functions with compact support
γ	trace operator
$\|\cdot\|_m$	Sobolev norm of order m
$\lvert\cdot\rvert_m$	Sobolev semi-norm of order m
$\|\cdot\|_\infty$	supremum norm
$\|\cdot\|_{m,h}$	mesh-dependent norm
ℓ_2	space of square-summable sequences
H'	dual space of H
$\langle\cdot,\cdot\rangle$	dual pairing
$\lvert\alpha\rvert$	$=\sum \alpha_i$, order of multiindex α
∂_i	partial derivative $\frac{\partial}{\partial x_i}$
∂^α	partial derivative of order α
D	(Fréchet) derivative
α	ellipticity constant
ν	exterior normal
∂_ν	$\partial/\partial\nu$, derivative in the direction of the exterior normal
∇f	$(\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_n)$
$\operatorname{div} f$	$\sum_{i=1}^n (\partial f_i/\partial x_i)$
S_h	finite element space
ψ_h	basis function in S_h
$\mathcal{T}_h$	partition of Ω
T	(triangular or quadrilateral) element in $\mathcal{T}_h$

T_{ref}	reference element
h_T, ρ_T	radii of circumscribed circle and incircle of T, respectively
κ	shape parameter of a partition
$\mu(T)$	area (volume) of T
$\mathcal{P}_t$	set of polynomials of degree $\leq t$
$\mathcal{Q}_t$	polynomial set (II.5.4) w.r.t. quadrilateral elements
$\mathcal{P}_{3,\text{red}}$	cubic polynomial without bubble function term
Π_{ref}	set of polynomials which are formed by the restriction of S_h to a (reference) element
s	$= \dim \Pi_{\text{ref}}$
Σ	set of linear functionals in the definition of affine families
$\mathcal{M}^k, \mathcal{M}^k_s, \mathcal{M}^k_{s,0}$	polynomial finite element spaces in L_2, H^{s+1} and H_0^{s+1}
$\mathcal{M}^1_{*,0}$	set of functions in $\mathcal{M}^1$ which are continuous at the midpoints of the sides and which satisfy zero boundary conditions in the same sense
RT_k	Raviart–Thomas element of degree k
I, I_h	interpolation operators on Π_{ref} and on S_h, respectively
A	stiffness or system matrix
$\delta_{..}$	Kronecker symbol
e	edge of an element
$\ker L$	kernel of the linear mapping L
$V^\perp$	orthogonal complement of V
V^0	polar of V
$\mathcal{L}$	Lagrange function
M	space of restrictions (for saddle point problems)
β	constant in the Brezzi condition
$H(\text{div}, \Omega)$	$:= \{v \in L_2(\Omega)^d;\ \text{div}\, v \in L_2(\Omega)\}$, $\Omega \in \mathbb{R}^d$
$L_{2,0}(\Omega)$	set of functions in $L_2(\Omega)$ with integral mean 0
B_3	cubic bubble functions
$\eta_{...}$	error estimator

Notation for the Method of Conjugate Gradients

∇f	gradient of f (column vector)
$\kappa(A)$	spectral condition number of the matrix A
$\sigma(A)$	spectrum of the matrix A
$\rho(A)$	spectral radius of the matrix A
$\lambda_{\min}(A)$	smallest eigenvalue of the matrix A
$\lambda_{\max}(A)$	largest eigenvalue of the matrix A
A^t	transpose of the matrix A
I	unit matrix
C	preconditioning matrix
g_k	gradient at the actual approximation x_k

d_k	direction of the correction in step k
V_k	$= \text{span}[g_0, \dots, g_{k-1}]$
$x'y$	Euclidean scalar product of the vectors x and y
$\|x\|_A$	$= \sqrt{x'Ax}$ (energy norm)
$\|x\|_\infty$	$= \max_i \|x_i\|$ (maximum norm)
T_k	k-th Chebyshev polynomial
ω	relaxation parameter

Notation for the Multigrid Method

$\mathcal{T}_\ell$	triangulation on the level ℓ
$S_\ell = S_{h_\ell}$	finite element space on the level ℓ
A_ℓ	system matrix on the level ℓ
N_ℓ	$= \dim S_\ell$
$\mathcal{S}$	smoothing operator
$r, \tilde{r}$	restrictions
p	prolongation
$x^{\ell,k,m}, u^{\ell,k,m}$	variable on the level ℓ in the k-th iteration step and in the m-th substep
ν_1, ν_2	number of presmoothings or postsmoothings, respectively
ν	$= \nu_1 + \nu_2$
μ	$= 1$ for V-cycle, $= 2$ for W-cycle
q	$= \ell_{\max}$
ψ_ℓ^j	j-th basis function on the level ℓ
ρ_ℓ	convergence rate of $\mathbf{MGM}_\ell$
ρ	$= \sup_\ell \rho_\ell$
$\vert\vert\vert \cdot \vert\vert\vert_s$	discrete norm of order s
β	measure of the smoothness of a function in S_h
$\mathcal{L}$	nonlinear operator
$\mathcal{L}_\ell$	nonlinear mapping on the level ℓ
$D\mathcal{L}$	derivative of $\mathcal{L}$
λ	homotopy parameter for incremental methods

Notation for Solid Mechanics

u	displacement
ϕ	deformation
id	identity mapping
C	$= \nabla\phi^T \nabla\phi$ Cauchy–Green strain tensor
E	strain
ε	strain in a linear approximation
t	Cauchy stress vector
T	Cauchy stress tensor
T_R	first Piola–Kirchhoff stress tensor
Σ_R	second Piola–Kirchhoff stress tensor

$\hat{T}$	$= \hat{T}(F)$ response function for the Cauchy stress tensor
$\hat{\Sigma}$	$= \hat{\Sigma}(F)$ response function for the Piola–Kirchhoff stress tensor
$\tilde{\Sigma}$	$\tilde{\Sigma}(F^T F) = \hat{\Sigma}(F)$
$\bar{T}$	$\bar{T}(FF^T) = \hat{T}(F)$
σ	stress in linear approximation
S^2	unit sphere in $\mathbb{R}^3$
$\mathbb{M}^3$	set of 3×3 matrices
$\mathbb{M}^3_+$	set of matrices in $\mathbb{M}^3$ with positive determinants
$\mathbb{O}^3$	set of orthogonal 3×3 matrices
$\mathbb{O}^3_+$	$= \mathbb{O}^3 \cap \mathbb{M}^3_+$
$\mathbb{S}^3$	set of symmetric 3×3 matrices
$\mathbb{S}^3_>$	set of positive definite matrices in $\mathbb{S}^3$
ι_A	$= (\iota_1(A), \iota_2(A), \iota_3(A))$, invariants of A
$\wedge$	vector product in $\mathbb{R}^3$
$\mathrm{diag}(d_1, \ldots, d_n)$	diagonal matrix with elements $d_1, \ldots, d_n$
λ, μ	Lamé constants
E	modulus of elasticity
ν	Poisson ratio
n	normal vector (different from Chs. II and III)
$\mathcal{C}$	$\sigma = \mathcal{C}\varepsilon$
$\hat{W}$	energy functional of hyperelastic materials
$\tilde{W}$	$\tilde{W}(F^T F) = \hat{W}(F)$
$\varepsilon : \sigma$	$= \sum_{ij} \varepsilon_{ij}\sigma_{ij}$
Γ_0, Γ_1	parts of the boundary on which u and$\sigma \cdot n$ are prescribed, respectively
Π	energy functional in the linear theory
$\nabla^{(s)}$	symmetric gradient
$as(\tau)$	skew-symmetric part of τ
$H^s(\Omega)^d$	$= [H^s(\Omega)]^d$
$H^1_\Gamma(\Omega)$	$:= \{v \in H^1(\Omega)R;\ v(x) = 0 \text{ for } x \in \Gamma_0\}$
$H(\mathrm{div}, \Omega)$	$:= \{\tau \in L_2(\Omega);\ \mathrm{div}\,\tau \in L_2(\Omega)\}$, τ is a vector or a tensor
$H(\mathrm{rot}, \Omega)$	$:= \{\eta \in L_2(\Omega)^2;\ \mathrm{rot}\,\eta \in L_2(\Omega)\},\ \Omega \subset \mathbb{R}^2$
$H^{-1}(\mathrm{div}, \Omega)$	$:= \{\tau \in H^{-1}(\Omega)^d;\ \mathrm{div}\,\tau \in H^{-1}(\Omega)\},\ \Omega \subset \mathbb{R}^d$
θ, γ, w	rotation, shear term, and transversal displacement of beams and plates
t	thickness of a beam, membrane, or plate
ℓ	length of a beam
$W_h, \Theta_h, \Gamma_h, Q_h$	finite element spaces in plate theory
π_h	L_2-projector onto Γ_h
R	restriction to Γ_h
P_h	L_2-projector onto Q_h

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Chapter I
Introduction

In dealing with partial differential equations, it is useful to differentiate between several types. In particular, we classify partial differential equations of second order as *elliptic, hyperbolic*, and *parabolic*. Both the theoretical and numerical treatment differ considerably for the three types. For example, in contrast with the case of ordinary differential equations where either initial or boundary conditions can be specified, here the type of equation determines whether initial, boundary, or initial-boundary conditions should be imposed.

The most important application of the finite element method is to the numerical solution of elliptic partial differential equations. Nevertheless, it is important to understand the differences between the three types of equations. In addition, we present some elementary properties of the various types of equations. Our discussion will show that for differential equations of elliptic type, we need to specify boundary conditions and not initial conditions.

There are two main approaches to the numerical solution of elliptic problems: *finite difference methods* and *variational methods*. The finite element method belongs to the second category. Although finite element methods are particularly effective for problems with complicated geometry, finite difference methods are often employed for simple problems, primarily because they are simpler to use. We include a short and elementary discussion of them in this chapter.