Graduate Texts in Mathematics

Serge Lang

Complex Analysis

Third Edition

复分析 第3版

Springer-Verlag
世界图书出版公司

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With 140 Illustrations

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

书　　名：Complex Analysis 3rd ed.

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中 译 名：复分析 第 3 版

出 版 者：世界图书出版公司北京公司

印 刷 者：北京世图印刷厂

发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）

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开　　本：24　　　印　张：20

出版年代：2003 年 6 月

书　　号：7-5062-6006-9 / O・395

版权登记：图字:01-2003-3770

定　　价：59.00 元

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Foreword

The present book is meant as a text for a course on complex analysis at the advanced undergraduate level, or first-year graduate level. The first half, more or less, can be used for a one-semester course addressed to undergraduates. The second half can be used for a second semester, at either level. Somewhat more material has been included than can be covered at leisure in one or two terms, to give opportunities for the instructor to exercise individual taste, and to lead the course in whatever directions strikes the instructor's fancy at the time as well as extra reading material for students on their own. A large number of routine exercises are included for the more standard portions, and a few harder exercises of striking theoretical interest are also included, but may be omitted in courses addressed to less advanced students.

In some sense, I think the classical German prewar texts were the best (Hurwitz–Courant, Knopp, Bieberbach, etc.) and I would recommend to anyone to look through them. More recent texts have emphasized connections with real analysis, which is important, but at the cost of exhibiting succinctly and clearly what is peculiar about complex analysis: the power series expansion, the uniqueness of analytic continuation, and the calculus of residues. The systematic elementary development of formal and convergent power series was standard fare in the German texts, but only Cartan, in the more recent books, includes this material, which I think is quite essential, e.g., for differential equations. I have written a short text, exhibiting these features, making it applicable to a wide variety of tastes.

The book essentially decomposes into two parts.

The *first part*, Chapters I through VIII, includes the basic properties of analytic functions, essentially what cannot be left out of, say, a one-semester course.

I have no fixed idea about the manner in which Cauchy's theorem is to be treated. In less advanced classes, or if time is lacking, the usual hand waving about simple closed curves and interiors is not entirely inappropriate. Perhaps better would be to state precisely the homological version and omit the formal proof. For those who want a more thorough understanding, I include the relevant material.

Artin originally had the idea of basing the homology needed for complex variables on the winding number. I have included his proof for Cauchy's theorem, extracting, however, a purely topological lemma of independent interest, not made explicit in Artin's original *Notre Dame* notes [Ar 65] or in Ahlfors' book closely following Artin [Ah 66]. I have also included the more recent proof by Dixon, which uses the winding number, but replaces the topological lemma by greater use of elementary properties of analytic functions which can be derived directly from the local theorem. The two aspects, homotopy and homology, both enter in an essential fashion for different applications of analytic functions, and neither is slighted at the expense of the other.

Most expositions usually include some of the global geometric properties of analytic maps at an early stage. I chose to make the preliminaries on complex functions as short as possible to get quickly into the analytic part of complex function theory: power series expansions and Cauchy's theorem. The advantages of doing this, reaching the heart of the subject rapidly, are obvious. The cost is that certain elementary global geometric considerations are thus omitted from Chapter I, for instance, to reappear later in connection with analytic isomorphisms (Conformal Mappings, Chapter VII) and potential theory (Harmonic Functions, Chapter VIII). I think it is best for the coherence of the book to have covered in one sweep the basic analytic material before dealing with these more geometric global topics. Since the proof of the general Riemann mapping theorem is somewhat more difficult than the study of the specific cases considered in Chapter VII, it has been postponed to the second part.

The *second and third parts* of the book, Chapters IX through XVI, deal with further assorted analytic aspects of functions in many directions, which may lead to many other branches of analysis. I have emphasized the possibility of defining analytic functions by an integral involving a parameter and differentiating under the integral sign. Some classical functions are given to work out as exercises, but the gamma function is worked out in detail in the text, as a prototype.

The chapters in Part II allow considerable flexibility in the order they are covered. For instance, the chapter on analytic continuation, including the Schwarz reflection principle, and/or the proof of the Riemann mapping theorem could be done right after Chapter VII, and still achieve great coherence.

As most of this part is somewhat harder than the first part, it can easily be omitted from a course addressed to undergraduates. In the same spirit, some of the harder exercises in the first part have been starred, to make their omission easy.

Comments on the Third Edition

I have rewritten some sections and have added a number of exercises. I have added some material on the Borel theorem and Borel's proof of Picard's theorem, as well as D.J. Newman's short proof of the prime number theorem, which illustrates many aspects of complex analysis in a classical setting. I have made more complete the treatment of the gamma and zeta functions. I have also added an Appendix which covers some topics which I find sufficiently important to have in the book. The first part of the Appendix recalls summation by parts and its application to uniform convergence. The others cover material which is not usually included in standard texts on complex analysis: difference equations, analytic differential equations, fixed points of fractional linear maps (of importance in dynamical systems), and Cauchy's formula for C^∞ functions. This material gives additional insight on techniques and results applied to more standard topics in the text. Some of them may have been assigned as exercises, and I hope students will try to prove them before looking up the proofs in the Appendix.

I am very grateful to several people for pointing out the need for a number of corrections, especially Wolfgang Fluch, Alberto Grunbaum, Bert Hochwald, Michal Jastrzebski, Ernest C. Schlesinger, A. Vijayakumar, Barnet Weinstock, and Sandy Zabell.

New Haven 1992 SERGE LANG

Prerequisites

We assume that the reader has had two years of calculus, and has some acquaintance with epsilon–delta techniques. For convenience, we have recalled all the necessary lemmas we need for continuous functions on compact sets in the plane. Section §1 in the Appendix also provides some background.

We use what is now standard terminology. A function

$$f: S \to T$$

is called **injective** if $x \neq y$ in S implies $f(x) \neq f(y)$. It is called **surjective** if for every z in T there exists $x \in S$ such that $f(x) = z$. If f is surjective, then we also say that f maps S **onto** T. If f is both injective and surjective then we say that f is **bijective**.

Given two functions f, g defined on a set of real numbers containing arbitrarily large numbers, and such that $g(x) \geqq 0$, we write

$$f \ll g \qquad \text{or} \qquad f(x) \ll g(x) \qquad \text{for} \quad x \to \infty$$

to mean that there exists a number $C > 0$ such that for all x sufficiently large, we have

$$|f(x)| \leqq Cg(x).$$

Similarly, if the functions are defined for x near 0, we use the same symbol $\ll$ for $x \to 0$ to mean that there exists $C > 0$ such that

$$|f(x)| \leqq Cg(x)$$

for all x sufficiently small (there exists $\delta > 0$ such that if $|x| < \delta$ then $|f(x)| \leqq Cg(x)$). Often this relation is also expressed by writing

$$f(x) = O(g(x)),$$

which is read: $f(x)$ is **big oh of** $g(x)$, for $x \to \infty$ or $x \to 0$ as the case may be.

We use $]a, b[$ to denote the **open** interval of numbers

$$a < x < b.$$

Similarly, $[a, b[$ denotes the half-open interval, etc.

Contents

PART ONE

Basic Theory

CHAPTER I

Complex Numbers and Functions

One of the advantages of dealing with the real numbers instead of the rational numbers is that certain equations which do not have any solutions in the rational numbers have a solution in real numbers. For instance, $x^2 = 2$ is such an equation. However, we also know some equations having no solution in real numbers, for instance $x^2 = -1$, or $x^2 = -2$. We define a new kind of number where such equations have solutions. The new kind of numbers will be called **complex** numbers.

I, §1. DEFINITION

The **complex numbers** are a set of objects which can be added and multiplied, the sum and product of two complex numbers being also a complex number, and satisfy the following conditions.

1. Every real number is a complex number, and if α, β are real numbers, then their sum and product as complex numbers are the same as their sum and product as real numbers.
2. There is a complex number denoted by i such that $i^2 = -1$.
3. Every complex number can be written uniquely in the form $a + bi$ where a, b are real numbers.
4. The ordinary laws of arithmetic concerning addition and multiplication are satisfied. We list these laws:

   If α, β, γ are complex numbers, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$, and

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma).$$

We have $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, and $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$.

We have $\alpha\beta = \beta\alpha$, and $\alpha + \beta = \beta + \alpha$.

If 1 is the real number one, then $1\alpha = \alpha$.

If 0 is the real number zero, then $0\alpha = 0$.

We have $\alpha + (-1)\alpha = 0$.

We shall now draw consequences of these properties. With each complex number $a + bi$, we associate the point (a, b) in the plane. Let $\alpha = a_1 + a_2 i$ and $\beta = b_1 + b_2 i$ be two complex numbers. Then

$$\alpha + \beta = a_1 + b_1 + (a_2 + b_2)i.$$

Hence addition of complex numbers is carried out "componentwise". For example, $(2 + 3i) + (-1 + 5i) = 1 + 8i$.

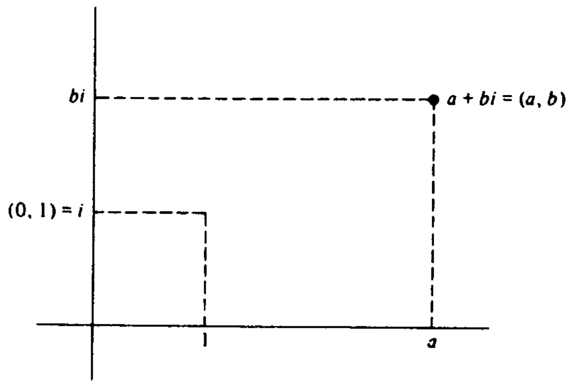

Figure 1

In multiplying complex numbers, we use the rule $i^2 = -1$ to simplify a product and to put it in the form $a + bi$. For instance, let $\alpha = 2 + 3i$ and $\beta = 1 - i$. Then

$$\begin{aligned} \alpha\beta = (2 + 3i)(1 - i) &= 2(1 - i) + 3i(1 - i) \\ &= 2 - 2i + 3i - 3i^2 \\ &= 2 + i - 3(-1) \\ &= 2 + 3 + i \\ &= 5 + i. \end{aligned}$$

Let $\alpha = a + bi$ be a complex number. We define $\bar{\alpha}$ to be $a - bi$. Thus if $\alpha = 2 + 3i$, then $\bar{\alpha} = 2 - 3i$. The complex number $\bar{\alpha}$ is called the