国外数学名著系列

（影印版）28

Peter Brass William Moser János Pach

Research Problems in Discrete Geometry

离散几何中的研究问题

科学出版社
www.sciencep.com

图字:01-2006-7387

图书在版编目(CIP)数据

离散几何中的研究问题=Research Problems in Discrete Geometry/(美)布拉斯(Brass, P.)等著.—影印版.—北京:科学出版社,2007
(国外数学名著系列)

ISBN 978-7-03-018292-0

Ⅰ.离… Ⅱ.布… Ⅲ.离散数学:几何学-研究-英文 Ⅳ.O18

中国版本图书馆 CIP 数据核字(2006)第 153198 号

责任编辑:范庆奎/责任印刷:安春生/封面设计:黄华斌

科学出版社 出版
北京东黄城根北街16号
邮政编码:100717
http://www.sciencep.com

中国科学院印刷厂 印刷

科学出版社发行 各地新华书店经销

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2007 年 1 月第 一 版 开本:B5(720×1000)
2007 年 1 月第一次印刷 印张:32 1/4
印数:1—3 500 字数:614 000

定价:82.00 元

(如有印装质量问题,我社负责调换〈科印〉)

中国科学院科学出版基金资助出版

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

You say you've got a real solution
Well, you know
We'd all love to see the plan
You ask me for a contribution
Well, you know
We are doing what we can
(John Lennon)

Preface

The forerunner of this book had a modest beginning in July 1977 at the Discrete Geometry Week (organized by H.S.M. Coxeter) in Oberwolfach, Germany. There, William Moser distributed a list of 14 problems that he called RPDG (Research Problems in Discrete Geometry). The problems had first appeared in a 1963 mimeographed collection of 50 problems proposed by Leo Moser (1921–1970) with the title "Poorly formulated unsolved problems in combinatorial geometry." Five new editions of RPDG appeared between 1977 and 1981, with hundreds of copies mailed to interested geometers; reviews of RPDG appeared in *Mathematics Magazine* 53 (1980) p. 189; *American Mathematical Monthly* 87 (1980) p. 236; *Zentralblatt für Mathematik* Zbl 528.52001 and *Mathematical Reviews* MR 84c:51003, MR 85h:52002. The 1986 edition of RPDG reported on the solution of several outstanding problems in earlier editions and was prepared with the collaboration of János Pach; the 1993 edition appeared as DIMACS Technical Report 93-32, 131 pp. We had hoped to publish a book soon thereafter. Indeed, Paul Erdős, the great problem proposer and collector, wrote a preface for that book in the expectation that it would soon be published. However, the book-writing project languished until 2000, when Peter Brass joined the project; his hard and careful work was instrumental in bringing the project to a conclusion. The book finally exists.

Many problems had to be left out, for in a subject with an active research community and a tradition of problem proposing it is natural that the number of open problems explodes over time. Our selection of problems is subjective, and many areas, such as art gallery problems, Helly-type questions, stochastic geometry, and problems about convex polytopes, are completely missing. We decided not to delay further, since a published incomplete book is more useful than an unpublished book (which would also be incomplete). Perhaps later in this century we will expand the collection in a second edition and report then that many current problems have been solved. Meanwhile, we invite the readers to submit their comments, corrections, and new problems to the site `http://www.math.nyu.edu/~pach/`. Whenever it was possible, we tried to give proper credit to the original problem proposers and problem solvers, but we have surely made many mistakes. We apologize for them, and we urge our readers to point out any error of this kind that they may discover.

Our aim all along has been to achieve a collection of research problems in discrete geometry containing a statement of each problem, an account of progress, and an up-to-date bibliography. It was meant to be a resource for everyone, but particularly for students and for young mathematicians, to help them in finding an interesting problem for research. Apart from the important open problems in the field, we have included a large number of less well known but beautiful questions whose solutions may not require deep methods. We wish the reader good luck in finding solutions.

We sincerely thank all those who helped us with encouragement, information, and corrections. These include Boris Aronov, Vojtech Bálint, Imre Bárány, András Bezdek, Károly Bezdek, Károly Böröczky Jr., Helmut Brass, Erik Demaine, Adrian Dumitrescu, Herbert Edelsbrunner, György Elekes, Christian Elsholtz, Gábor Fejes Tóth, Eli Goodman, Ronald Graham, Branko Grünbaum, Heiko Harborth, Martin Henk, Aladár Heppes, Ferran Hurtado, Dan Ismailescu, Gyula Károlyi, Arnfried Kemnitz, Wlodzimierz Kuperberg, Endre Makai, Rados Radoičić, Andrej Raĭgorodskĭi, Imre Z. Ruzsa, Micha Sharir, Alexander Soifer, József Solymosi, Konrad Swanepoel, Gábor Tardos, Csaba D. Tóth, Géza Tóth, Pavel Valtr, Katalin Vesztergombi, Jörg Wills, Chuanming Zong, and two students, Zheng Zhang and Mehrbod Sharifi. We apologize to those whose names have inadvertently been left out. We thank Marion Blake, David Kramer, Ina Lindemann, Paula Moser, and Mark Spencer for valuable editorial assistance, and Danielle Spencer for her help in preparing the cover design. We thank the mathematics libraries at the Free University Berlin, the Technical University Braunschweig, the Mathematische Forschungsinstitut Oberwolfach and at Courant Institute, New York University; our work would not have been possible without access to these excellent libraries. We also thank all our friends who obtained literature for us that we could not get ourselves.

This book is dedicated to Gisela and Helmut Brass and to Heiko Harborth (respectively parents and advisor of Peter Brass); to Beryl Moser and Leo Moser (respectively wife and brother of William Moser); to Klára and Zsigmond Pál Pach (parents of János Pach).

City College New York — Peter Brass
McGill University — William Moser
City College New York, NYU, and Rényi Institute — János Pach

Preface to an Earlier Version of RPDG

My friend Leo Moser (1921–1970) was an avid creator, collector, and solver of problems in number theory and combinatorics. At the 1963 Number Theory Conference in Boulder, Colorado, he distributed mimeographed copies of his list of fifty problems, which he called "Poorly formulated unsolved problems in combinatorial geometry." Although some parts of this collection have been reproduced several times, the entire list in its original form appeared in print only recently (*Discrete Applied Math.* **31** (1991), 201–225).

After Leo Moser's death, his brother Willy put together his *Research Problems in Discrete Geometry (RPDG)*, which was based on some questions proposed by Leo and was first distributed among the participants of the Discrete Geometry week in Oberwolfach, July 1977. This collection has been revised and largely extended by W. Moser and J. Pach. It has become an excellent resource book of fascinating open problems in combinatorial and discrete geometry which had nine different editions circulating in more than a thousand copies. In the last fifteen years it has reached virtually everybody interested in the field, and has generated a lot of research. In addition to the many new questions, a number of important but badly forgotten problems have also been publicized in these collections. They include Heilbronn's (now famous) triangle problem and my old questions about the distribution of distances among n points in the plane, just to mention two areas where much progress has been made recently. The present book is an updated "final" version of a large subset of the problems that appeared in the previous informal editions of *Research Problems in Discrete Geometry.* The authors have adopted a very pleasant style that allows the reader to get not only a feel for the problems but also an overview of the field.

And now let me say a few words about discrete geometry. As a matter of fact, I cannot even give a reasonable definition of the subject. Perhaps it is not inappropriate to recall the following old anecdote. Some years ago, when pornography was still illegal in America, a judge was asked to define pornography. He answered: "I cannot do this, but I sure can recognize it when I see it."

Perhaps discrete geometry started with the feud between Newton and Gregory about the largest number of solid unit ball spheres that can be placed to touch a "central" unit ball sphere. Newton believed this number to be twelve, while Gregory believed it was thirteen. This controversy was settled in Newton's favor only late in the last century. Even today little is known about similar problems in higher dimensions, although these questions were kept alive by the nineteenth century crystallographers and have created a lot of interest among physicists and biologists.

Minkowski's book *Geometrie der Zahlen* (1896) opened a new and important chapter in mathematics. It revealed some surprising connections between number theory and convex geometry, particularly between diophantine approximation and packing problems. This branch of discrete geometry was developed in books by Cassels (*An Introduction to the Geometry of Numbers*), Lekkerkerker (*Geometry of Numbers*), Coxeter (*Regular Polytopes*), and L. Fejes Tóth (*Lagerungen in der Ebene, auf der Kugel und im Raum*). "Alles Konvexe interessiert mich," said Minkowski, and I share his feeling.

Another early source is Sylvester's famous "orchard problem." In 1893 he also raised the following question: Given n points in the plane, not all on a line, can one always find a line passing through exactly two points? This problem remained unsolved and was completely forgotten before I rediscovered it in 1933. I was reading the Hilbert and Cohn-Vossen book (*Anschauliche Geometrie*) when the question occurred to me, and I thought it was new. It looked innocent, but to my surprise and annoyance I was unable to resolve it. However, I immediately realized that an affirmative answer would imply that any set of n noncollinear points in the plane determines at least n connecting lines. A couple of days later, Tibor Gallai came up with an ingenious short proof which turned out to be the first solution of Sylvester's problem. This was the starting point of many fruitful investigations about the incidence structure of sets of points and lines, circles, etc. Recently, these results have attracted a lot of attention, because they proved to be relevant in computational geometry.

In 1931, E. Klein observed that from any five points in the plane in general position one can choose four that determine a convex quadrilateral, and she asked whether the following generalization was true: For any $k \geq 4$ there exists an integer n_k such that any n_k-element set of points in general position in the plane contains the vertex set of a convex k-gon. Szekeres and I managed to establish this result; for the first proof we needed, and Szekeres rediscovered, Ramsey's theorem! Our paper raised many fascinating new questions which, I think, gave a boost to the development of combinatorial geometry and extremal combinatorics. A large variety of problems of this kind is discussed in the books of Hadwiger and Debrunner (*Combinatorial Geometry in the Plane*, translated and extended by Klee), Grünbaum (*Convex Polytopes*), Croft, Falconer, and Guy (*Unsolved Problems in Geometry*), and in the collection of my papers (*The Art of Counting*). I hope that the reader will forgive me that the above sketch of the recent history of combinatorial and discrete geometry is very subjective and, of course, overemphasizes my own contribution to the field.

There are certain areas of mathematics where individual problems are less important. However, I feel that problems play a very important role in elementary number theory and geometry. Hilbert and Hermann Weyl had the same opinion, but many eminent mathematicians disagree. I cannot decide who is right, but I am certainly on the side of Grünbaum in his old controversy with Dieudonné, who claimed that geometry is "dead." We are convinced that if a subject is rich in simple and fascinating unsolved problems, then it has a great future! The present collection of research problems by Moser and Pach proves beyond doubt the richness of discrete geometry.

I wish the reader good luck with the solutions!

Budapest, May 1991

Paul Erdős

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Contents

0. Definitions and Notations

In this short chapter, we have collected some definitions and notations that are used in many places in this book. All of the concepts are quite standard; we list them for completeness and to explain the notation.

A set C is *convex* if for any two points $p, q \in C$ the entire line segment pq is also contained in C. A set C is *star-shaped* if for some point $p \in C$ and all points $q \in C$, the entire line segment pq is also contained in C. A set is a *convex body* if it is convex, compact and has nonempty interior. In general, a *body* is a set homeomorphic to a ball. Let $\mathbb{R}^d$ stand for the d-dimensional Euclidean space. In $\mathbb{R}^d$, the d-dimensional ball of radius r around the origin is denoted by $B^d(r)$, and the unit ball $B^d(1)$ by B^d. The two-dimensional ball B^2 is called a *circle* (we try to avoid the word "disk," which is often used in the literature for plane convex bodies).

Two bodies are *nonoverlapping* if they do not have an interior point in common, and they *touch* each other if they are nonoverlapping but have a common boundary point.

Some important functions defined for convex bodies C are the *volume* $\mathrm{Vol}(C)$; the *diameter* $\mathrm{diam}(C)$, which is the maximum distance between two points of C; the *width* $\mathrm{width}(C)$, which is the smallest distance of two parallel hyperplanes such that C lies in the slab between them; and the *inradius* and *circumradius*, which are the radii of the largest ball contained in C and the smallest ball containing C.

The Minkowski sum $X + Y$ of two sets is the set $\{x + y \mid x \in X, y \in Y\}$. Similarly, $\lambda X = \{\lambda x \mid x \in X\}$ denotes a scaled copy of X, and $-X = -1X = \{-x \mid x \in X\}$ denotes a copy of X reflected through the point 0. These operations depend on the choice of the origin 0, but the results are the same up to translation, and the operations should be viewed as acting on translation equivalence classes.

The *Hausdorff distance* of two compact sets $X, Y \subset \mathbb{R}^d$ is defined by

$$d^{\substack{\text{Haus-}\\\text{dorff}}}(X,Y) = \max\Big(\sup_{x\in X}\inf_{y\in Y} d_{\mathrm{eucl}}(x,y)\ ,\ \sup_{y\in Y}\inf_{x\in X} d_{\mathrm{eucl}}(x,y)\Big).$$

An alternative description using Minkowski sums is

$$d^{\substack{\text{Haus-}\\\text{dorff}}}(X,Y) = \min\Big\{\lambda \geq 0 \,\Big|\, X + \lambda B^d \supseteq Y \text{ and } Y + \lambda B^d \supseteq X\Big\}.$$

Some important classes of set mappings are *translations*, *homotheties*, *congruences*, *similarities*, and *affine maps*. A translate of a set $X \subset \mathbb{R}^d$ is a set $X + t$, $t \in \mathbb{R}^d$, a homothetic copy is a scaled translate $\lambda X + t$ with $\lambda > 0$. *Negative homothetic copies* with $\lambda < 0$ are allowed only where it is explicitly stated. A congruence is an isometry (reflections are allowed), and a similarity is a scaled congruence. An affinity is a nondegenerate linear transformation followed by a translation.

A *symmetry* of a set is a congruence that maps the set onto itself. A set X is called *centrally symmetric* about the origin if $X = -X$. In general, X is centrally symmetric about the point (vector) t if $X = -X + 2t$.

A *lattice* Λ can be viewed in two ways, as a set of translations or as a set of points, the *lattice points.*

As a set of translations, Λ is the set of all linear combinations of the elements of a basis of the space with integer coefficients, which is the group of translations generated by this basis. For any d linearly independent vectors $u_1, \ldots, u_d$ in d-dimensional space, let $\Lambda = \Lambda(u_1, \ldots, u_d)$ denote the lattice generated by them, so

$$\Lambda = \{m_1 u_1 + \cdots + m_d u_d \mid m_1, \ldots, m_d \in \mathbb{Z}\}.$$

A *fundamental domain* of Λ is a closed set whose translates by the elements of Λ tile the space.

As a set of points, Λ is the orbit of any point p under the above set of translations, that is, the set $\{p + m_1 u_1 + \cdots + m_d u_d \mid m_1, \ldots, m_d \in \mathbb{Z}\}$. Thus, the lattice points of Λ are a translation equivalence class of point sets.

The parallelepiped P induced by the 2^d vertices of the form $m_1 u_1 + \cdots + m_d u_d$, where $m_i \in \{0, 1\}$ for every i, is called a *fundamental parallelepiped* of the lattice. The same lattice can of course be generated in many different ways and, therefore, has infinitely many different fundamental parallelepipeds. All of them are fundamental domains of the lattice, seen as a group of translations, and therefore all of them have the same volume

$$\mathrm{Vol}(P) = |\det(u_1, \ldots, u_d)|.$$

The *density* of a lattice is defined as the reciprocal of this determinant $|\det(u_1, \ldots, u_d)|$. This number is equal to the limit of the number of lattice points in the ball $B^d(r)$ divided by $\mathrm{Vol}(B^d(r))$, as r tends to infinity.

Any lattice similar to the planar lattice generated by two adjacent sides of a square or equilateral triangle is called a *square lattice* or a *triangular lattice*, respectively.

A finite-dimensional *normed space*, also called *Minkowski space*, is a finite-dimensional linear space X equipped with metric which is translation-invariant ($d(p, q) = d(p + t, q + t)$ for translations t) and homogeneous ($d(\lambda p, 0) = \lambda d(p, 0)$ for $\lambda > 0$). Since $d(p, q) = d(p - q, 0)$, this metric is completely described by the distance of every point x from the origin, which is called the *norm* $\|x\| = d(x, 0)$. A normed space can be characterized by