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Introduction to Banach Spaces: Analysis and Probability

Volume I

DANIEL LI
HERVÉ QUEFFÉLEC

This two-volume text provides a complete overview of the theory of Banach spaces, emphasising its interplay with classical and harmonic analysis (particularly Sidon sets) and probability. The authors give a full exposition of all results, as well as numerous exercises and comments to complement the text and aid graduate students in functional analysis. The book will also be an invaluable reference volume for researchers in analysis.

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166

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Volume I

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QUEFFÉLEC

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Introduction to Banach Spaces: Analysis and Probability

Volume 1

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Jean-Pierre Kahane

Preface

This book is dedicated to the study of Banach spaces.

While this is an introduction, because we trace this study back to its origins, it is indeed a "specialized course",[1] in the sense that we assume that the reader is familiar with the general notions of Functional Analysis, as taught in late undergraduate or graduate university programs. Essentially, we assume that the reader is familiar with, for example, the first ten chapters of Rudin's book, Real and Complex Analysis (RUDIN 2); QUEFFÉLEC–ZUILY would also suffice.

It is also a "specialized course" because the subjects that we have chosen to study are treated in depth.

Moreover, as this is a textbook, we have taken the position to completely prove all the results "from scratch" (i.e. without referring within the proof to a "well-known result" or admitting a difficult auxiliary result), by including proofs of theorems in Analysis, often classical, that are not usually taught in French universities (as, for example, the interpolation theorems and the Marcel Riesz theorem in Chapter 7 of Volume 1, or Rademacher's theorem in Chapter 1 of Volume 2). The exceptions are a few results at the end of the chapters, which should be considered as complementary, and are not used in what follows.

We have also included a relatively lengthy first chapter introducing the fundamental notions of Probability.

As we have chosen to illustrate our subject with applications to "thin sets" coming from Harmonic Analysis, we have also included in Volume 1 an Annex devoted to compact Abelian groups.

This makes for quite a thick book,[2] but we hope that it can therefore be used without the reader having to constantly consult other texts.

[1] The French version of this book appeared in the collection "Cours Spécialisés" of the Société Mathématique de France.

[2] However, divided into two parts in the English version.

We have emphasized the aspects linked to Analysis and Probability; in particular, we have not addressed the geometric aspects at all; for these we refer, for example, to the classic DAY, to BEAUZAMY or to more specialized books such as BENYAMINI–LINDENSTRAUSS, DEVILLE–GODEFROY–ZIZLER or PISIER 2.

We have hardly touched on the study of operators on Banach spaces, for which we refer to TOMCZAK-JAEGERMANN and to PISIER 2; DIESTEL–JARCHOW–TONGE and PIETSCH–WENZEL are also texts in which the part devoted to operators is more important. DUNFORD–SCHWARTZ remains a very good reference.

Even though Probability plays a large role here, this is not a text about Probability in Banach spaces, a subject perfectly covered in LEDOUX–TALAGRAND.

Probability and Banach spaces were quick to get on well together. Although the study of random variables with values in Banach spaces began as early as the 1950s (R. Fortet and E. Mourier; we also cite Beck [1962]), their contribution to the study of Banach spaces themselves only appeared later, for example, citing only a few, Bretagnolle, Dacunha-Castelle and Krivine [1966], and Rosenthal [1970] and [1973]. However, it was only with the introduction of the notions of type and cotype of Banach spaces (Hoffmann-Jørgensen [1973], Maurey [1972 b] and [1972 c], Maurey and Pisier [1973]) that they proved to be intimately linked with Banach spaces.

Moreover, Probability also arises in Banach spaces by other aspects; notably it allows the derivation of the very important Dvoretzky's theorem (Chapter 1 of Volume 2), thanks to the concentration of measure phenomenon, a subject still highly topical (see the recent book of M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs **89**, AMS, 2001), dating back to Paul Lévy, and whose importance for Banach spaces was seen by Milman at the beginning of the 1970s.

We will also use Probability in a third manner, through the method of selectors, due to Erdös around 1955,[3] and afterwards used heavily by Bourgain, which allows us to make random constructions.

For all that, we do not limit ourselves to the probabilistic aspects; we also wish to show how the study of Banach spaces and of classical analysis interact (the construction by Davie, in Chapter 2 of Volume 2, of Banach spaces without the approximation property is typical in this regard); in particular we have concentrated on the application to thin sets in Harmonic Analysis.

Even if we have privileged these two points of view, we have nonetheless tried to give a global view of Banach spaces (with the exception of the

[3] Actually, this method traces back at least to Cramér [1935] and [1937].

geometric aspect, as already mentioned), with the concepts and fundamental results up through the end of the 1990s.

We point out that an interesting survey of what was known by the mid 1970s was given by Pełczyński and Bessaga [1979].

This book is divided into 14 chapters, preceded by a preliminary chapter and accompanied by an Annex. The first volume contains the first eight chapters, including the preliminary chapter and the Annex; the second volume contains the six remaining chapters. Moreover, it also contains three surveys, by G. Godefroy, O. Guédon and G. Pisier, on the major results and directions taken by Banach space theory since the publication of the French version of this book (2004), as well as an original paper of L. Rodríguez-Piazza on Sidon sets.

Each chapter is divided into sections, numbered by Roman numerals in capital letters (**I**, **II**, **III** etc.), and each section into subsections, numbered by Arabic numerals (**I.1** etc.). The theorems, propositions, corollaries, lemmas, definitions are numbered successively in the interior of each section; for example in Chapter 5 of Volume 1, Section IV they thus appear successively in the form: Proposition IV.1, Corollary IV.2, Definition IV.3, Theorem IV.4, Lemma IV.5, ignoring the subsections. If we need to refer to one chapter from another, the chapter containing the reference will be indicated.

At the end of each chapter, we have added comments. Certain of these cite complementary results; others provide a few indications of the origin of the theorems in the chapter. We have been told that "this is a good occasion to antagonize a good many colleagues, those not cited or incorrectly cited." We have done our best to correctly cite, in the proper chronological order, the authors of such and such result, of such and such proof. No doubt errors or omissions have been made; they are only due to the limits of our knowledge. When this is the case, we ask forgiveness in advance to the interested parties. We make no pretension to being exhaustive, nor to be working as historians. These indications should only be taken as incitements to the reader to refer back to the original articles and as complements to the contents of the course.

The chapters end with exercises. Many of these propose proofs of recent, and often important, results. In any case, we have attempted to decompose the proofs into a number of questions (which we hope are sufficient) so that the reader can complete all the details; just to make sure, in most cases we have indicated where to find the corresponding article or book.

The citations are presented in the following manner: if it concerns a book, the name of the author (or the authors) is given in small capitals, for example BANACH, followed by a number if there are several books by this author: RUDIN 3; if it concerns an article or contribution, it is cited by the name of the author or authors, followed in brackets by the year of publication, followed possibly by a lower-case letter: Salem and Zygmund [1954], James [1964 a].

We now come to a more precise description of what will be found in this book.

In the Preliminary Chapter, we quickly present some useful properties concerning the weak topology $w = \sigma(E, E^*)$ of a Banach space E and the weak* topology $w^* = \sigma(X^*, X)$ in a dual space X^*. Principally, we will prove the Eberlein–Šmulian theorem about weakly compact sets and the Krein–Milman theorem on extreme points. We then provide some information about filters and countable ordinals.

Chapter 1 of Volume 1 is intended for readers who have never been exposed to Probability Theory. With the exception of Section V concerning martingales, which will not be used until Chapter 7, its contents are quite elementary and very classical; let us say that they provide "Probability for Analysts." Moreover, in this book, we use little more than (but intensively!) Gaussian random variables (occasionally stable variables), and the Bernoulli or Rademacher random variables. The reader could refer to BARBE–LEDOUX or to REVUZ.

Section III provides the theorems of Kolmogorov for the convergence of series of independent random variables, and the equivalence theorem of Paul Lévy.

In Section IV, we show Khintchine's inequalities, which, even if elementary, are of capital importance for Analysis. We also find here the majorant theorem (Theorem IV.5) which will be very useful throughout the book.

Section V, a bit delicate for a novice reader of Probability, remains quite classical; we introduce martingales and prove Doob's theorems about their convergence.

In Chapter 2 (Volume 1) we begin the actual study of Banach spaces. We treat the Schauder bases, which provide a common and very practical tool.

After having shown in Section II that the projections associated with a basis are continuous and given a few examples (canonical bases of c_0, ℓ_p, Haar basis in $L^p(0,1)$, Schauder basis of $\mathcal{C}([0,1])$), we prove that the space $\mathcal{C}([0,1])$ is universal for the separable spaces, i.e. any separable Banach space is isometric to a subspace of $\mathcal{C}([0,1])$.

In Section III, we see how the use of bases, or more generally of basic sequences, allows us to obtain structural results; notably, thanks to the Bessaga–Pełczyński selection theorem, to show that any Banach space contains a subspace with a basis. We next show a few properties of the spaces c_0 and ℓ_p. Finally, we see how the spaces possessing a basis behave with respect to duality; this leads to the notions of shrinking bases and boundedly complete bases and to the corresponding structure theorems of James.

In Chapter 3 (Volume 1), we study the properties of unconditional convergence (i.e. commutative convergence) of series in Banach spaces.

After having given different characterizations of this convergence (Proposition II.2) and showed the Orlicz–Pettis theorem (Theorem II.3) in Section II, we introduce in Section III the notion of unconditional basis, and show, in particular, that the sequences of centered independent random variables are basic and unconditional in the spaces $L^p(\mathbb{P})$.

In Section IV, we study in particular the canonical basis of c_0, and prove the theorems of Bessaga and Pełczyński which, on one hand, characterize the presence of c_0 within a space by the existence of a scalarly summable sequence that is not summable, and, on the other hand, state that a dual space containing c_0 must contain ℓ_∞.

In Section V, we describe the James structure theorems characterizing, among the spaces having an unconditional basis, those containing c_0, or ℓ_1, or those that are reflexive.

All of the above work was done before 1960 and is now very classical.

In Section VI, we prove the Gowers dichotomy theorem, stating that every Banach space contains a subspace with an unconditional basis or a hereditarily indecomposable subspace (that is, none of its infinite-dimensional closed subspaces can be decomposed as a direct sum of infinite-dimensional closed subspaces). In addition, we provide a sketch of the proof of the homogeneous subspace theorem: every infinite-dimensional space that is isomorphic to all of its infinite-dimensional subspaces is isomorphic to ℓ_2.

In Chapter 4 (Volume 1), we study random variables with values in Banach spaces.

Section II essentially states that the properties of convergence in probability, almost surely, and in distribution, seen in Chapter 1 in the scalar case can be generalized "as such" for the vector-valued case. Prokhorov's theorem (Theorem II.9) characterizes the families of relatively compact probabilities on a Polish space. The conditional expectation, more delicate to define than in the scalar case, is introduced, as well as martingales; the vectorial version of Doob's theorem (Theorem II.12) then easily follows from the scalar case.

In Section III we describe the important symmetry principle, also known as the Paul Lévy maximal inequality, which allows us to obtain the equivalence theorem for series of independent Banach-valued random variables between convergence in distribution, almost sure and in probability.

The contraction principle of Section IV will be of fundamental importance for all that follows; in its quantitative version, it essentially states that for a real (respectively complex) Banach space E, the sequences of independent centered random variables in $L^p(E)$, $1 \leqslant p < +\infty$, are unconditional basic sequences with constant 2 (respectively 4).

In Section V, we generalize the scalar Khintchine inequalities to the vectorial case (Kahane inequalities); the proof is much more difficult than for the scalar case. These inequalities will turn out to be very important when we define the type and the cotype of Banach spaces (Chapter 5). The proof of the Kahane inequalities uses probabilistic arguments; in Subsection V.3, we will see how the use of the Walsh functions allowed Latała and Oleskiewicz, thanks to a hypercontractive property of certain operators (Proposition V.6), to obtain, in the case "$L^1 - L^2$," the best constant for these inequalities (Theorem V.4).

Chapter 5 (Volume 1) introduces the fundamental notions of type and cotype of Banach spaces.

It is now common practice to define these using Rademacher variables, but it is often more interesting to use Gaussian variables, notably for their invariance under rotation. We thus begin, in Section II, by providing some complements of Probability; we first define Gaussian vectors, and show their invariance under rotation (Proposition II.8); we take advantage of this to present the vectorial version of the central limit theorem, which we will use in Chapter 4 of Volume 2. We next prove the existence of p-stable variables, also to be used in Chapter 4 of Volume 2, and present the classical theorems of Schönberg on the kernels of positive type, and of Bochner, which characterizes the Fourier transforms of measures.

As notions of type and cotype are local, i.e. only involving the structure of finite dimensional subspaces, we give a few words in Section III to ultraproducts and to spaces finitely representable within another; we prove the local reflexivity theorem of Lindenstrauss and Rosenthal, stating, more or less, that the finite-dimensional subspaces of the bidual are almost isometric to subspaces of the space itself.

In Section IV, we define the type and cotype, give a few examples (type and cotype of L^p spaces, cotype 2 of the dual of a C^*-algebra), a few properties, and see how these notions behave with duality; this leads to the notion of K-convexity. We also show that in spaces having a non-trivial type, respectively cotype, we can, in the definition, replace the Rademacher variables by Gaussian variables (Theorem IV.8).

In Section V, we prove Kwapień's theorem, stating that a space is isomorphic to a Hilbert space if, and only if, it has at the same time type 2 and cotype 2; for this we first study the operators that factorize through a Hilbert space.

In Section VI, we present a few applications, and in particular show how to obtain the classical theorems of Paley and Carleman (Theorem VI.2).

In Chapter 6 (Volume 1), we will study a very important notion, that of a p-summing operator, brought out by Pietsch in 1967, and which soon afterward allowed Lindenstrauss and Pełczyński to highlight the importance of Grothendieck's theorem, which, even though proven in the mid 1950s, had not until then been properly understood.

We begin with an introduction showing that the 2-summing operators on a Hilbert space are the Hilbert–Schmidt operators.

In Section II, after having given the definition and pointed out the ideal property possessed by the space of p-summing operators, we prove the Pietsch factorization theorem, stating that the p-summing operators $T\colon X \to Y$ are those that factorize by the canonical injection (or rather its restriction to a subspace) of a space $\mathcal{C}(K)$ in $L^p(K, \mu)$, where K is a compact (Hausdorff) space and μ a regular probability measure on K; in particular the 2-summing operators factorize through a Hilbert space. It easily follows that the p-summing operators are weakly compact and are Dunford–Pettis operators. We next prove, thanks to Khintchine's inequalities, a theorem of Pietsch and Pełczyński stating that the Hilbert–Schmidt operators on a Hilbert space are not only 2-summing, but even 1-summing.

In Section III, we show Grothendieck's inequality (Theorem III.3), stating that scalar matrix inequalities are preserved when we replace the scalars by elements of a Hilbert space, losing at most a constant factor K_G, called the Grothendieck constant. We then prove Grothendieck's theorem: every operator of a space $L^1(\mu)$ into a Hilbert space is 1-summing. The proof is "local," meaning that it involves only the finite-dimensional subspaces; in passing we also show that the finite-dimensional subspaces of L^p spaces can be embedded, $(1+\varepsilon)$-isomorphically, within spaces of sequences ℓ_p^N of finite dimension N. We then give the dual form of this theorem: every operator of a space $L^\infty(\nu)$ into a space $L^1(\mu)$ is 2-summing.

In Section IV, we present a number of results, originally proven in different ways, that can easily be obtained using the properties of p-summing operators (note that these do not depend on Grothendieck's theorem, contrary to what might be suggested by the order of the presentation): the Dvoretzky–Rogers theorem (every infinite-dimensional space contains at least one sequence unconditionally convergent but not absolutely convergent), John's theorem (the Banach–Mazur distance of every space of dimension n to the space ℓ_2^n is $\leqslant \sqrt{n}$), and the Kadeč–Snobar theorem (in any Banach space, there exists, on every subspace of dimension n, a projection of norm $\leqslant \sqrt{n}$). We then see that Grothendieck's theorem allows us to show that every normalized unconditional basis of ℓ_1 or of c_0 is equivalent to their canonical basis (this is also true for ℓ_2, but this case is easy).

Finally, Section V is devoted to Sidon sets (see Definition V.1). The fundamental example is that of Rademacher variables in the dual of the Cantor group $\Omega = \{-1, +1\}^{\mathbb{N}}$; another example is that of powers of 3 in $\mathbb{Z}$. We prove a certain number of properties, functional, arithmetical and combinatorial, demonstrating the "smallness" of Sidon sets; we show in passing the classical inequality of Bernstein. Grothendieck's theorem allows us to show that a set Λ is Sidon if and only if the space $\mathcal{C}_\Lambda$ is isomorphic to ℓ_1. We next present a theorem that is very important for the study of Sidon sets, Rider's theorem (Theorem V.18), which involves, instead of the uniform norm of polynomials, another norm $[\![\,.\,]\!]_R$, obtained by taking the expectation of random polynomials constructed by multiplying the coefficients by independent Rademacher variables. This allows us to obtain Drury's theorem (Theorem V.20), stating that the union of two Sidon sets is again a Sidon set, and the fact, due to Pisier, that Λ is a Sidon set as soon as $\mathcal{C}_\Lambda$ is of cotype 2; for this last result, we need to replace, in the norm $[\![\,.\,]\!]_R$, the Rademacher variables by Gaussian variables, and are led to show a property of integrability of Gaussian vectors, due to Fernique (Theorem V.26), a Gaussian version of the Khintchine–Kahane inequalities, which will also be useful in Chapter 6 of Volume 2.

In Chapter 7 (Volume 1), we present a few properties of the spaces L^p. In Section II, we study the space L^1. After having defined the notion of uniform integrability, we give a condition for a *sequence* of functions to be uniformly integrable (the Vitali–Hahn–Saks theorem), which allows us to deduce that the spaces $L^1(m)$ are weakly sequentially complete. We then characterize the weakly compact subsets of L^1 as being the weakly closed and uniformly integrable subsets (the Dunford–Pettis theorem). We conclude this section by showing that L^1 is not a subspace of a space with an unconditional basis. We will continue the study of L^1 in Chapter 4 of Volume 2; more specifically, we will examine the structure of its reflexive subspaces.

In Section III, we will see that the trigonometric system forms a basis of $L^p(0,1)$ for $p > 1$. This is in fact an immediate consequence of the Marcel Riesz theorem, stating that the Riesz projection, or the Hilbert transform, is continuous on L^p for $p > 1$; most of Section III is hence devoted to the proof of this result. We have chosen not to prove it directly, but to reason by interpolation, allowing us to show in passing the Marcinkiewicz theorem, at the origin of real interpolation, as well as Kolmogorov's theorem stating that the Riesz projection is of weak type $(1, 1)$ (Theorem III.6). We conclude this section with a result of Orlicz (Corollary III.9) stating that the unconditional convergence of a series in L^p, for $1 \leqslant p \leqslant 2$, implies the convergence of the sum of the squares of the norms, implying that the trigonometric system is unconditional only for L^2.