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INTRODUCTION TO MODERN PRIME NUMBER THEORY

BY

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PREFACE

This book takes the reader as far as Vinogradoff's theorem, that every sufficiently large odd positive integer can be represented as a sum of three primes. It assumes nothing of the theory of numbers that is not given in Hardy and Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), hereafter quoted as H.-W.

My main purpose in writing this book was to enable those mathematicians who are not specialists in the theory of numbers to learn some of its non-elementary results and methods without too great an effort.

Chapter 1 deals with a refinement of the prime number theorem, obtained by de la Vallée Poussin in 1899 (*Mém. cour. Acad. R. Belg.* **59**, 1), three years after the discovery by him and Hadamard of the prime number theorem itself. The method of proof used here is essentially due to Landau.

The main result of Chapter 2, Theorem 55, with its emphasis on uniformity in k, was stated and proved in 1936 by Walfisz (*Math. Z.* **40**, 598, Hilfssatz 3), but the difficulty in obtaining it had then been removed by Siegel (*Acta Arith.* **1** (1935), 83–6), who had discovered a property of Dirichlet's L functions which led at once to Theorem 48. The method used in §§ 2·5–2·6 is taken from the theory of groups. This theory is not assumed, but the reader who finds §§2·5–2·7 difficult may be referred to an alternative proof of Theorem 28: Landau, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), **1**, Satz 134.

The method of Chapter 3, excluding Theorem 56, is due to Hardy and Littlewood (*Acta Math., Stockh.*, **44** (1923), 1–70). Vinogradoff, in obtaining his famous result (*Rec. Math.* T. 2 (44), 2 (1937), 179–95), built on foundations laid by them. At the same time Theorem 56, which is due to him, is a very substantial contribution.

I am indebted to my colleague Mr. H. Kestelman for valuable advice and criticism, and to Mr. R. C. Wellard for checking part of the manuscript and correcting a mistake.

T. ESTERMANN

London, August 1951

REMARKS ON NOTATION

Throughout this book, the following letters denote the following types of number:

h, j, l, m, n = integers;
k, q = positive integers;
p = primes;
$t, u, v, x, y, \sigma, \theta$ = real numbers;
M, ϵ, δ = positive numbers;
s, w, z = complex numbers.

The real and imaginary parts of s are, as usual, denoted by σ and t respectively.

$C_1, C_2, \ldots$ are suitable (sufficiently large) positive absolute constants.

$\int_z^w$ denotes an integral taken along the straight line from z to w.

$[x]$ denotes the greatest integer less than or equal to x.

$e(z)$ is an abbreviation for $e^{2\pi i z}$.

CONTENTS

INTRODUCTION TO MODERN PRIME NUMBER THEORY

CHAPTER 1

THE RIEMANN ZETA FUNCTION AND A REFINEMENT OF THE PRIME NUMBER THEOREM

1·1. The prime number theorem states that $\pi(m)$, the number of primes not exceeding m, is asymptotic to $m/\log m$. Our object is to obtain a better approximation to $\pi(m)$, and we shall show that

$$\pi(m) = \sum_{n=2}^{m} \frac{1}{\log n} + O(me^{-c\sqrt{\log m}}), \tag{1}$$

where c is a suitable positive constant. This is a refinement of the prime number theorem, for it is easily seen that

$$\sum_{n=2}^{m} \frac{1}{\log n} \sim \frac{m}{\log m}.$$

The numerical value of c is unimportant. In fact, (1) is true for any positive constant c, and this can be shown by a slight modification of the method used here. Still better results in this direction can be proved by more complicated methods.

The proof of (1) is mostly analytical. Only the last step is elementary, and consists of a straightforward argument which deduces (1) from the formula

$$\psi(m) = m + O(me^{-c\sqrt{\log m}}), \tag{2}$$

where

$$\psi(m) = \sum_{n=1}^{m} \Lambda(n), \tag{3}$$

$\Lambda(p^k) = \log p$, and $\Lambda(n) = 0$ if n is not of the form p^k. The analytical part of the proof depends on certain properties of the Riemann zeta function, originally defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad (\sigma > 1). \tag{4}$$

We shall extend this definition by analytic continuation up to the imaginary axis. The further analytic continuation of the zeta function into the left-hand half-plane, though well known, will not be given in this book, as it is not needed for our purpose. The deepest property of the zeta function used here is that

$$\zeta(s) \neq 0 \quad \{\sigma > 1 - 1/(C_1 \log |t|),\ |t| > C_1\}. \tag{5}$$

1·2. The details are as follows. In order to obtain the analytic continuation of $\zeta(s)$, we consider the functions

$$f_n(s) = n^{-s} - \int_n^{n+1} u^{-s}\,du \quad (n = 1, 2, \ldots). \tag{6}$$

It is obvious that
$$\zeta(s) = \sum_{n=1}^{\infty} f_n(s) + \frac{1}{s-1} \tag{7}$$

if $\sigma > 1$. Also
$$f_n(s) = \int_n^{n+1} (n^{-s} - u^{-s})\,du$$

and

$$|n^{-s} - u^{-s}| = \left| \int_n^u s v^{-s-1}\,dv \right| \leq |s| \int_n^{n+1} v^{-\sigma-1}\,dv \quad (n \leq u \leq n+1),$$

so that
$$|f_n(s)| \leq |s| \int_n^{n+1} v^{-\sigma-1}\,dv. \tag{8}$$

We use the term '*locally uniformly at* s_0' for 'uniformly in some circle about s_0' and the term '*locally uniformly in* S' (where S is a set of points) for 'locally uniformly at all points of S'. It follows from (8) that $\sum_{n=1}^{\infty} f_n(s)$ converges locally uniformly in the half-plane $\sigma > 0$. Hence we may *define* $\zeta(s)$ for $0 < \sigma \leq 1$, $s \neq 1$, by (7), thus obtaining the desired analytic continuation and

THEOREM 1. *$\zeta(s)$ is regular for $\sigma > 0$, except at $s = 1$, where it has a simple pole with residue* 1.

We also deduce from (7) and (8) that

$$\left|\zeta(s) - \frac{1}{s-1}\right| \leqslant |s| \int_1^\infty v^{-\sigma-1}\,dv = \frac{|s|}{\sigma} \quad (\sigma > 0,\ s \neq 1). \tag{9}$$

The next theorem, which belongs to the elementary theory of numbers, will be used here repeatedly.

THEOREM 2. *If $f(n)$ is multiplicative, and $\sum_{n=1}^{\infty} |f(n)|$ converges, then*

$$\sum_{n=1}^{\infty} f(n) = \prod_p \sum_{m=0}^{\infty} f(p^m).$$

This follows from H.-W., Theorem 286, on putting $s = 0$ and noting that $f(1) = 1$ for any multiplicative function $f(n)$ which does not vanish identically. Conversely, Theorem 2 implies H.-W., Theorem 286, since n^{-s} is a multiplicative function of n.

THEOREM 3. *Let $\sigma > 1$. Then*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

This follows from (4) and Theorem 2. We deduce that

$$\zeta(s) \neq 0 \quad (\sigma > 1). \tag{10}$$

THEOREM 4. *Let $\sigma > 1$. Then*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_p \frac{\log p}{p^s - 1}.$$

This follows from Theorem 3 since the series converges locally uniformly in the half-plane $\sigma > 1$. We deduce that

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad (\sigma > 1) \tag{11}$$

(cf. H.-W., Theorem 294, where s is, however, restricted to real values).

1·3. The next theorem, which is almost trivial, will help us to establish two further properties of the zeta function needed in our proof of (5).

THEOREM 5. *Let* $|z| = 1$. *Then* $\mathbf{R}(3+4z+z^2) \geqslant 0$.

Proof. Putting $z = x+iy$, we have $x^2+y^2 = 1$ and hence $\mathbf{R}(3+4z+z^2) = 3+4x+x^2-y^2 = 2+4x+2x^2 = 2(1+x)^2 \geqslant 0.$

The two properties referred to are as follows:

THEOREM 6. *Let* $u > 1$. *Then*

$$\mathbf{R}\left\{3\frac{\zeta'(u)}{\zeta(u)}+4\frac{\zeta'(u+iv)}{\zeta(u+iv)}+\frac{\zeta'(u+2iv)}{\zeta(u+2iv)}\right\} \leqslant 0 \tag{12}$$

and

$$|\zeta^3(u)\,\zeta^4(u+iv)\,\zeta(u+2iv)| \geqslant 1. \tag{13}$$

Proof. By (11),

$$3\frac{\zeta'(u)}{\zeta(u)}+4\frac{\zeta'(u+iv)}{\zeta(u+iv)}+\frac{\zeta'(u+2iv)}{\zeta(u+2iv)} = -\sum_{n=1}^{\infty}\Lambda(n)\,n^{-u}a_n,$$

where $a_n = 3+4n^{-iv}+n^{-2iv}$, so that, by Theorem 5, $\mathbf{R}a_n \geqslant 0$. This proves (12). Also, by Theorem 3,

$$\zeta(s) = \exp\sum_p \log(1-p^{-s})^{-1} = \exp\sum_p\sum_{m=1}^{\infty}\frac{p^{-ms}}{m}$$

$$= \exp\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}n^{-s} \quad (\sigma > 1),$$

so that

$$|\zeta^3(u)\,\zeta^4(u+iv)\,\zeta(u+2iv)| = \exp\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}n^{-u}\mathbf{R}a_n$$

with a_n defined as before. Since $\mathbf{R}a_n \geqslant 0$, this proves (13).

1·4. The next four theorems belong to the theory of functions. The third will be the main analytical tool in the proof of (5). The fourth will only be used in the next chapter.

THEOREM 7. *Let* $r>1$,

$$f(z)=\sum_{n=1}^{\infty} b_n z^n \quad (|z|<r), \quad \textit{and} \quad \mathbf{R}f(z)\leqslant M \quad (|z|=1).$$

Then $$|b_n|\leqslant 2M \quad (n=1,2,\ldots).$$

Proof. Putting $b_n=|b_n|e^{i\theta_n}$, we have

$$\mathbf{R}f(e^{i\theta})=\sum_{n=1}^{\infty}|b_n|\cos(\theta_n+n\theta),$$

and this series converges uniformly, so that

$$\int_0^{2\pi}\mathbf{R}f(e^{i\theta})\,d\theta=0$$

and $$\int_0^{2\pi}\mathbf{R}f(e^{i\theta})\cos(\theta_n+n\theta)\,d\theta=\pi|b_n| \quad (n=1,2,\ldots),$$

which implies that

$$\pi|b_n|=\int_0^{2\pi}\mathbf{R}f(e^{i\theta})\{1+\cos(\theta_n+n\theta)\}\,d\theta$$
$$\leqslant\int_0^{2\pi}M\{1+\cos(\theta_n+n\theta)\}\,d\theta=2\pi M,$$

and the result follows.

THEOREM 8. *Let* $r>1$, *let* $g(z)$ *be regular for* $|z|<r$, *and let* $g(z)\neq 0$ $(|z|<r)$ *and* $|g(z)/g(0)|\leqslant e^M$ $(|z|=1)$. *Then* $|g'(0)/g(0)|\leqslant 2M$.

This follows from Theorem 7 with

$$f(z)=\int_0^z\frac{g'(w)}{g(w)}\,dw \quad \text{and} \quad b_n=f^{(n)}(0)/n!,$$

which implies that $$g'(0)/g(0)=b_1.$$

THEOREM 9. *Let* $f(z)$ *be regular, and* $|f(z)/f(0)|\leqslant e^M$, *for* $|z|\leqslant 2$; *let* $0<a\leqslant 1$,

$$f(z)\neq 0 \quad (|z|\leqslant 1,\ \mathbf{R}z>0), \tag{14}$$

and let $f(z)$ *have a zero of order* h *at* $z=-a$. *Then*

$$-\mathbf{R}\{f'(0)/f(0)\}\leqslant 2M-h/a.$$

Proof. Let the zeros of $f(z)$ within and on the circle $|z| = 1$ be $z_1, z_2, \ldots, z_l$, of orders $k_1, k_2, \ldots, k_l$ respectively, and put

$$g(z) = f(z) \prod_{m=1}^{l} (z - z_m)^{-k_m}.$$

Then $g(z)$ is regular for $|z| \leqslant 2$, and there is a number $r > 1$ such that $g(z) \neq 0 \quad (|z| < r)$. Also

$$\left|\frac{g(z)}{g(0)}\right| = \left|\frac{f(z)}{f(0)}\right| \prod_{m=1}^{l} \left(\frac{|z_m|}{|z - z_m|}\right)^{k_m} \leqslant \left|\frac{f(z)}{f(0)}\right| \leqslant e^M \quad (|z| = 2),$$

and hence, by the maximum modulus theorem,

$$|g(z)/g(0)| \leqslant e^M \quad (|z| = 1).$$

From this and Theorem 8 it follows that

$$-\mathbf{R}\{g'(0)/g(0)\} \leqslant |g'(0)/g(0)| \leqslant 2M,$$

which means that $-\mathbf{R}\left\{\dfrac{f'(0)}{f(0)} + \displaystyle\sum_{m=1}^{l} \frac{k_m}{z_m}\right\} \leqslant 2M,$

so that
$$-\mathbf{R}\frac{f'(0)}{f(0)} \leqslant 2M + \sum_{m=1}^{l} k_m \mathbf{R}\frac{1}{z_m}.$$

Now, by (14), all the terms $k_m \mathbf{R}(1/z_m)$ are negative or 0, and $-h/a$ is one of them. Hence the result.

THEOREM 10. *Let $f(z)$ be regular, and $|f(z)/f(0)| \leqslant e^M$, for $|z| \leqslant 2$; let $|a| \leqslant 1$, $|b| \leqslant 1$, $a \neq b$, $f(a) = f(b) = 0$, and let (14) hold. Then $-\mathbf{R}\{f'(0)/f(0)\} \leqslant 2M + \mathbf{R}(1/a) + \mathbf{R}(1/b)$.*

This can be proved in the same way as Theorem 9.

1·5. We are now in a position to prove a theorem which implies (5). For convenience, we put

$$t^* = \max(|t|, 100) \tag{15}$$

and
$$\eta(s) = (s-1)\zeta(s) \quad (s \neq 1), \quad \eta(1) = 1. \tag{16}$$

THEOREM 11. *$\zeta(s)$ has no zeros in the set of points D given by $\sigma > 1 - 1/(4000 \log t^*)$.*

Proof. Suppose that the theorem is false. Then there are numbers σ_0 and t_0 such that

$$\zeta(\sigma_0 + it_0) = 0 \tag{17}$$

and
$$\sigma_0 > 1 - 1/(4000 \log t_0^*), \tag{18}$$

where
$$t_0^* = \max(|t_0|, 100). \tag{19}$$

It follows from (10) and (17) that

$$\sigma_0 \leqslant 1, \tag{20}$$

and from this and (9), (17), and (18) that $|t_0| \geqslant \frac{3}{4}$. Hence, if $1 < u < 2$ and $|s - u - it_0| \leqslant \frac{1}{4}$, then $|s-1| \geqslant |t| \geqslant |t_0| - \frac{1}{4} \geqslant \frac{1}{2}$, $|s| \leqslant 3 + t_0^*$, and $\sigma > \frac{3}{4}$, so that, by (9) and (19),

$$|\zeta(s)| \leqslant 6 + \tfrac{4}{3}t_0^* \leqslant \tfrac{3}{2}t_0^* \quad (1 < u < 2,\ |s - u - it_0| \leqslant \tfrac{1}{4}). \tag{21}$$

Similarly $$|\zeta(s)| \leqslant 3t_0^* \quad (1 < u < 2,\ |s - u - 2it_0| \leqslant \tfrac{1}{4}). \tag{22}$$

Also, if $1 < u$ and $|s - u| \leqslant \frac{1}{4}$, then

$$\frac{\sigma^2}{|s|^2} \geqslant \frac{\sigma^2}{|s|^2} - \left(\frac{\sigma}{|s|} - \frac{|s|}{u}\right)^2 = 1 - \frac{|s-u|^2}{u^2} > \frac{15}{16}$$

($\frac{15}{16}$ is the square of the cosine of half the angle subtended at the origin by the circle $|s - 1| = \frac{1}{4}$), so that

$$|s|/\sigma < \sqrt{(16/15)} < 31/30.$$

Hence, by (16) and (9),

$$|\eta(s)| \leqslant 1 + \tfrac{31}{30}|s-1| < \tfrac{4}{3} \quad (1 < u < \tfrac{21}{20},\ |s-u| \leqslant \tfrac{1}{4}). \tag{23}$$

Now let
$$u = 1 + 1/(800 \log t_0^*) \tag{24}$$

and
$$f(z) = \eta^3(u + \tfrac{1}{8}z)\, \zeta^4(u + it_0 + \tfrac{1}{8}z)\, \zeta(u + 2it_0 + \tfrac{1}{8}z). \tag{25}$$

Then, by (23), (21), (22), and (19),

$$|f(z)| \leqslant (\tfrac{4}{3})^3 (\tfrac{3}{2} t_0^*)^4 \, 3t_0^* < t_0^{*6} \quad (|z| \leqslant 2). \tag{26}$$

Also, putting

$$g(w) = \zeta^3(u+w)\,\zeta^4(u+it_0+w)\,\zeta(u+2it_0+w), \tag{27}$$

we have, by (25) and (16),

$$f(z) = (u-1+\tfrac{1}{8}z)^3 g(\tfrac{1}{8}z),$$

which implies that
$$f(0) = (u-1)^3 g(0) \tag{28}$$

and
$$\frac{f'(0)}{f(0)} = \frac{3}{8(u-1)} + \frac{1}{8}\frac{g'(0)}{g(0)}. \tag{29}$$

Now, by (27) and Theorem 6, $|g(0)| \geqslant 1$ and $\mathbf{R}\{g'(0)/g(0)\} \leqslant 0$. Hence, by (24), (28), and (29),

$$|f(0)| \geqslant (800 \log t_0^*)^{-3} \tag{30}$$

and
$$\mathbf{R}\{f'(0)/f(0)\} \leqslant 300 \log t_0^*. \tag{31}$$

Also $x/\log x$ increases with x when $x \geqslant e$. Hence

$$t_0^*/\log t_0^* \geqslant 100/\log 100 > 20,$$

and hence, by (30), $|f(0)| > (40t_0^*)^{-3} > t_0^{*-6}$. From this and (26) it follows that

$$|f(z)/f(0)| \leqslant t_0^{*12} \quad (|z| \leqslant 2). \tag{32}$$

Now, by (17) and (25), $f(z)$ has a zero at $z = 8(\sigma_0 - u)$, and, denoting the order of this zero by h, we have

$$h \geqslant 4. \tag{33}$$

Putting $M = 12 \log t_0^*$ and $a = 8(w - \sigma_0)$, we can now verify that all the hypotheses of Theorem 9 are satisfied. In fact, $f(z)$ is regular for $|z| \leqslant 2$ by (25), (16), and Theorem 1, since $u > 1$ and $|t_0| \geqslant \frac{3}{4}$; $|f(z)/f(0)| \leqslant e^M (|z| \leqslant 2)$ by (32); $0 < a \leqslant 1$ by (24), (20), (18), and (19); (14) follows from (25), (16), and (10) since $u > 1$, and we have just seen that $f(z)$ has a zero of order h at $z = -a$. Hence, by Theorem 9, (31), and (33),

$$0 \leqslant 2M - \frac{4}{a} + 300 \log t_0^* = 324 \log t_0^* - \frac{1}{2(u-\sigma_0)}.$$

From this and (24) and (18) it follows that

$$324 \geqslant \frac{1}{2(u-\sigma_0)\log t_0^*} > \frac{1}{2(800^{-1}+4000^{-1})} = \frac{1000}{3}.$$

This is a contradiction; so Theorem 11 is proved.

THEOREM 12. *$\eta'(s)/\eta(s)$ is regular in the set of points D of Theorem* 11.

This follows from (16) and Theorems 1 and 11.

1·6. We shall deduce that $\eta'(s)/\eta(s) = O(\log^3 t^*)$ in a suitable subset of D. First, however, we have to prove another theorem from the theory of functions.

THEOREM 13. *Let $0 < r_1 < r_2$, let $g(z)$ be regular for $|z| < r_2$ and let $g(0) = 0$ and $\mathbf{R}g(z) \leqslant M$ $(|z| = r_1)$. Then*

$$|g'(z)| \leqslant 2Mr_1(r_1 - |z|)^{-2} \quad (|z| < r_1).$$

Proof. Let $f(z) = g(r_1 z)$ and $r = r_2/r_1$. Then there are numbers $b_1, b_2, \ldots$ such that all the hypotheses of Theorem 7 are satisfied. Hence, if $|z| < r_1$, we have

$$|g'(z)| = \frac{1}{r_1}\left|f'\left(\frac{z}{r_1}\right)\right| = \frac{1}{r_1}\left|\sum_{n=1}^{\infty} nb_n\left(\frac{z}{r_1}\right)^{n-1}\right| \leqslant \frac{1}{r_1}\sum_{n=1}^{\infty} n\,|b_n|\left(\frac{|z|}{r_1}\right)^{n-1}$$

$$\leqslant \frac{2M}{r_1}\sum_{n=1}^{\infty} n\left(\frac{|z|}{r_1}\right)^{n-1} = 2Mr_1(r_1 - |z|)^{-2},$$

which proves the theorem.

THEOREM 14. *Let* $1 - 1/(10000 \log t^*) \leqslant \sigma < 2$. *Then*

$$|\eta'(s)/\eta(s)| \leqslant C_2 \log^3 t^*.$$

Proof. Let $r_1 = 1 + 1/(5000 \log t^*)$, $r_2 = 1 + 1/(4500 \log t^*)$, and

$$g(z) = \int_{2+it}^{2+it+z} \frac{\eta'(w)}{\eta(w)}\, dw. \tag{34}$$

Then it is easily seen that any point w for which $|w - (2+it)| < r_2$ is a point of D (the set of points defined in Theorem 11). From this and Theorem 12 and (34) it follows that $g(z)$ is regular for $|z| < r_2$. Also, by (34), $e^{g(z)} = \eta(2+it+z)/\eta(2+it)$. Now, by (16), (9), and (15),

$$|\eta(2+it+z)| \leqslant 1 + \frac{(4+|t|)^2}{2-r_1} < 2t^{*2} \quad (|z| \leqslant r_1),$$

and, by (16) and Theorem 3,

$$\begin{aligned} |\eta(2+it)|^{-1} &\leqslant |\zeta(2+it)|^{-1} \\ &\leqslant \prod_p (1+p^{-2}) < \prod_p (1-p^{-2})^{-1} = \zeta(2) < 2. \end{aligned}$$

Hence $e^{\mathbf{R}g(z)} < 4t^{*2} < t^{*3}$ $(|z| = r_1)$, and the hypotheses of Theorem 13 are satisfied with $M = 3 \log t^*$. It follows that

$$\begin{aligned} |\eta'(s)/\eta(s)| &= |g'(s-2-it)| = |g'(\sigma-2)| \\ &\leqslant 2Mr_1(r_1 - 2 + \sigma)^{-2} \leqslant 2Mr_1(10000 \log t^*)^2 < 10^9 \log^3 t^*, \end{aligned}$$

which proves the theorem.

1·7. Our next task is to prove (2). The connexion between $\zeta(s)$ and $\psi(m)$ may be expressed in the formula

$$\psi(m) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{(m+\frac{1}{2})^s}{s} \frac{\zeta'(s)}{\zeta(s)}\, ds \quad (a > 1).$$

We shall neither prove nor use this formula. Instead, we consider the integral

$$\int_{a-ib}^{a+ib} \frac{(m+\frac{1}{2})^s}{s} \frac{\zeta'(s)}{\zeta(s)}\, ds,$$

which, for suitable values of a and b, provides a good enough approximation to $-2\pi i\psi(m)$. Then we replace $\zeta'(s)/\zeta(s)$ by $-1/(s-1)+\eta'(s)/\eta(s)$, using (16), and deduce from Theorem 12 and Cauchy's theorem that

$$\int_{a-ib}^{a+ib} \frac{(m+\frac{1}{2})^s}{s} \frac{\eta'(s)}{\eta(s)} ds = \int_C \frac{(m+\frac{1}{2})^s}{s} \frac{\eta'(s)}{\eta(s)} ds,$$

where C is a suitable broken line. Finally we obtain an inequality for the last integral from Theorem 14. These are the main steps in the proof of (2).

For convenience, we put

$$E(x) = \begin{cases} 1 & (x>1), \\ 0 & (0<x<1). \end{cases} \tag{35}$$

Then, by (3),

$$\psi(m) = \sum_{n=1}^{\infty} E\left(\frac{m+\frac{1}{2}}{n}\right) \Lambda(n) \quad (m = 1, 2, \ldots). \tag{36}$$

THEOREM 15. *Let $a>0$, $b>0$, $x>0$, and $x \neq 1$. Then*

$$\left| \int_{a-ib}^{a+ib} \frac{x^s}{s} ds - 2\pi i E(x) \right| \leqslant \frac{2x^a}{b\,|\log x|}.$$

Proof. Suppose, first, that $x>1$. Then it easily follows from the theorem of residues that

$$J_1 + \int_{a-ib}^{a+ib} \frac{x^s}{s} ds + J_2 = 2\pi i,$$

where
$$J_1 = \int_{-\infty-ib}^{a-ib} \frac{x^s}{s} ds, \quad J_2 = \int_{a+ib}^{-\infty+ib} \frac{x^s}{s} ds.$$

Now
$$|J_1| = \left| \int_{-\infty}^{a} \frac{x^{\sigma-ib}}{\sigma-ib} d\sigma \right| \leqslant \int_{-\infty}^{a} \frac{x^\sigma}{b} d\sigma = \frac{x^a}{b\log x}.$$

Similarly, $|J_2| \leqslant x^a/(b\log x)$, and the result follows in this case.