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SET VALUED MAPPINGS WITH APPLICATIONS IN NONLINEAR ANALYSIS

Ravi P. Agarwal and Donal O'Regan

Set Valued Mappings with Applications in Nonlinear Analysis

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Preface

This book is a collection of research articles related to the mathematical analysis of multifunctions. By a set valued map $F : X \to 2^Y$ we simply mean a map that assigns to each $x \in X$ a subset $F(x) \subseteq Y$. The theory of set valued maps is a beautiful mixture of analysis, topology and geometry. Over the last thirty years or so there has been a huge interest in this area of research. This is partly due to the rich and plentiful supply of applications in such diverse fields as for example Biology, Control theory and Optimization, Economics, Game theory and Physics. This book titled 'set valued mappings with applications in nonlinear analysis' contains 29 research articles from leading mathematicians in this area from around the world. Topological methods in the study of nonlinear phenomena is the central theme. As a result the chapters were selected accordingly and no attempt was made to cover every area in this vast field. The topics covered in this book can be grouped in the following major areas: integral inclusions, ordinary and partial differential inclusions, fixed point theorems, boundary value problems, variational inequalities, game theory, optimal control, abstract economics, and nonlinear spectra.

In particular the theory of set valued maps is used in the chapters of Agarwal, Meehan and O'Regan, Andres, Candito, Kamenski and Nistri, Kryszewski, Matzakos and Papageorgiou, and Palmucci and Papalini to present results for differential and integral inclusions in various settings. The Baire category method is used by De Blasi and Pianigiani to discuss existence problems for partial differential inclusions. Structure of solution sets is addressed by Agarwal and O'Regan, and Obukhovskii and Zecca. The chapter of Matzakos, Papageorgiou and Yannakakis contains results on optimal control for nonlinear parabolic partial differential equations. Many new fixed point theorems for set valued maps are contained in the contributions of Agarwal and O'Regan, Daffer and Kaneko, Frigon, Morales, Ricceri, and Takahashi. Nonlinear spectral theory is discussed by Appell, Conti and Santucci, random fixed point theory by Shahzad, and fuzzy mappings by Cho, Shim, Huang and Kang. In a long survey chapter Milojević presents new results in the theory of A-proper maps. Variational inequalities are discussed in the long survey article of Chowdhury and Tarafdar, and in the chapters of Isac, Tarafdar and Yuan, and Park. Maximal element principles are presented in the contributions of Ding, and Isac and Yuan. Applications of fixed point theory in abstract economies and game theory appear in the chapter of Tan and Wu. Some interesting fixed point algorithms are contained in the chapters of Reich and Zaslavski, and Verma.

We wish to express our appreciation to all the contributors. Without their cooperation this book would not have been possible.

Ravi P Agarwal
Donal O'Regan

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1. Positive L^p and Continuous Solutions for Fredholm Integral Inclusions

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Abstract: In this chapter a multivalued version of Krasnoselski's fixed point theorem in a cone is used to discuss the existence of $C[0,T]$ and $L^p[0,T]$ solutions to the nonlinear integral inclusion $y(t) \in \int_0^T k(t,s)\, f(s,y(s))\, ds$. Throughout we will assume k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to 2^{\mathbf{R}}$.

1. INTRODUCTION

In this chapter we present new results which guarantee that the Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s) f(s,y(s)) ds \tag{1.1}$$

has a positive solution $y \in L^p[0,T]$, $1 \le p < \infty$, or has a nonnegative solution $y \in C[0,T]$. Throughout this chapter $T > 0$ is fixed, k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to 2^{\mathbf{R}}$; here $2^{\mathbf{R}}$ denotes the family of nonempty subsets of $\mathbf{R}$. It is only recently [6] that a general theory has been developed which guarantees that the operator equation, $y(t) = \int_0^T k(t,s) g(s,y(s))\, ds$ for a.e. $t \in [0,T]$, has a positive solution $y \in L^p[0,T]$ (note by a positive solution we mean $y(t) > 0$ for a.e. $t \in [0,T]$); here g: $[0,T] \times \mathbf{R} \to \mathbf{R}$ is single valued. In Section 2 using the 1991 paper of Cellina *et al.* [3] we are able to establish criteria which guarantees that (1.1) has a positive solution $y \in L^p[0,T]$. Section 3 discusses $C[0,T]$ solutions to (1.1); the results here improve those in [1].

The main idea in this chapter relies on the multivalued analogue [1] of Krasnoselski's fixed point theorem in a cone. Let $E = (E, \|\cdot\|)$ be a Banach space and $C \subseteq E$. For $\rho > 0$ let

$$\Omega_\rho = \{x \in E : \|x\| < \rho\} \quad \text{and} \quad \partial\Omega_\rho = \{x \in E : \|x\| = \rho\}.$$

Theorem 1.1: *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone and let $\|\cdot\|$ be increasing with respect to C. Also r, R are constants with $0 < r < R$. Suppose $A\colon \overline{\Omega_R} \cap C \to K(C)$ (here $K(C)$ denotes the family of nonempty, convex, compact subsets of C) is an upper semicontinuous, compact map and assume* one *of the following conditions*

(*A*) $\|y\| \leq \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap C$ *and* $\|y\| > \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap C$

or

(*B*) $\|y\| > \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_R \cap C$ *and* $\|y\| \leq \|x\|$ *for all* $y \in A(x)$ *and* $x \in \partial\Omega_r \cap C$

hold. Then A has a fixed point in $C \cap (\overline{\Omega_R} \backslash \Omega_r)$.

2. $L^p[0, T]$ SOLUTIONS

In this section we discuss the nonlinear Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s) f(s, y(s)) ds \quad \text{a.e. } t \in [0,T], \tag{2.1}$$

where $k\colon [0,T] \times [0,T] \to \mathbf{R}$ and $f\colon [0,T] \times \mathbf{R} \to K(\mathbf{R})$. We would like to know what conditions one requires on k and f in order that the inclusion (2.1) has a positive solution $y \in L^p[0,T]$, where $1 \leq p < \infty$. Here by a positive solution y we mean $y(t) > 0$ for a.e. $t \in [0,T]$. Throughout this section $\|\cdot\|_q$ denotes the usual norm on L^q for $1 \leq q \leq \infty$.

Theorem 2.1: *Let $k\colon [0,T] \times [0,T] \to \mathbf{R}$ and $f\colon [0,T] \times \mathbf{R} \to K(\mathbf{R})$ and suppose the following conditions hold:*

$$\text{the map } u \mapsto f(t,u) \text{ is upper semicontinuous for a.e. } t \in [0,T]; \tag{2.2}$$

$$\begin{gathered}\text{the graph of } f \text{ belongs to the } \sigma\text{-field } \mathcal{L} \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R}) \\ (\text{here } \mathcal{L} \text{ denotes the Lebesgue } \sigma\text{-field on } [0,T] \text{ and } \mathcal{B}(\mathbf{R} \times \mathbf{R}) \\ = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \text{ is the Borel } \sigma\text{-field in } \mathbf{R} \times \mathbf{R});\end{gathered} \tag{2.3}$$

$$\begin{gathered}\exists p_2, 1 \leq p_2 < \infty, a_1 \in L^{p_2}[0,T] \text{ and } a_2 > 0 \text{ a constant, with} \\ |f(t,y)| = \sup\{|z| : z \in f(t,y)\} \leq a_1(t) + a_2 |y|^{\frac{p}{p_2}} \\ \text{for a.e. } t \in [0,T] \text{ and all } y \in \mathbf{R};\end{gathered} \tag{2.4}$$

$$(t,s) \mapsto k(t,s) \text{ is measurable}; \tag{2.5}$$

$$\begin{gathered}\exists\, 0 < M \leq 1, k_1 \in L^p[0,T], k_2 \in L^{p_1}[0,T], \text{ here } \tfrac{1}{p_1} + \tfrac{1}{p_2} = 1, \text{ such that} \\ 0 < k_1(t), k_2(t) \text{ a.e. } t \in [0,T] \text{ and } M k_1(t) k_2(s) \leq k(t,s) \leq k_1(t) k_2(s) \\ \text{a.e. } t \in [0,T], \text{ a.e. } s \in [0,T];\end{gathered} \tag{2.6}$$

$$\text{for a.e. } t \in [0,T] \text{ and all } y \in (0,\infty),\ u > 0 \text{ for all } u \in f(t,y); \tag{2.7}$$

$$\begin{aligned}&\exists\, q \in L^{p_2}[0,T] \text{ and } \psi\colon [0,\infty) \to [0,\infty),\ \psi(u) > 0 \text{ for } u > 0,\\ &\text{continuous and nondecreasing with for a.e. } t \in [0,T] \text{ and } y > 0,\\ &u \geq q(t)\psi(y) \text{ for all } u \in f(t,y);\end{aligned} \tag{2.8}$$

$$\exists\, \alpha > 0 \quad \textit{with} \quad 1 < \frac{\alpha}{2^{\frac{p_2-1}{p_2}} \|k_1\|_p \|k_2\|_{p1} \left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2}\, \alpha^p \right)^{\frac{1}{p_2}}} \tag{2.9}$$

and

$$\exists\, \beta > 0,\ \beta \neq \alpha \quad \textit{with} \quad 1 > \frac{\beta}{M\|k_1\|_p \int_0^T k_2(s)q(s)\psi(a(s)\beta)ds}, \tag{2.10}$$

where

$$a(t) = M\frac{k_1(t)}{\|k_1\|_p}. \tag{2.11}$$

Then (2.1) has at least one positive solution $y \in L^p[0,T]$ and either

(A) $0 < \alpha < \|y\|_p < \beta$ *and* $y(t) \geq a(t)\alpha$ *a.e.* $t \in [0,T]$ *if* $\alpha < \beta$

or

(B) $0 < \beta < \|y\|_p < \alpha$ *and* $y(t) \geq a(t)\beta$ *a.e.* $t \in [0,T]$ *if* $\beta < \alpha$

holds.

Proof: Let $E = (L^p[0,T], \|\cdot\|_p)$ and

$$C = \{y \in L^p[0,T] : y(t) \geq a(t)\|y\|_p \text{ a.e. } t \in [0,T]\}.$$

It is easy to see that $C \subseteq E$ is a cone. Next let $A = K \circ N_f\colon C \to 2^E$, where the linear integral (single valued) operator K is given by

$$Ky(t) = \int_0^T k(t,s)y(s)\,ds,$$

and the multivalued Nemytskij operator N_f is given by

$$N_f u = \{y \in L^{p_2}[0,T] : y(t) \in f(t,u(t)) \text{ a.e. } t \in [0,T]\}.$$

Remark 2.1: Note A is well defined since if $x \in C$ then (2.2)–(2.4) and [3] guarantee that $N_f x \neq \emptyset$.

We first show $A\colon C \to 2^C$. To see this let $x \in C$ and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t \in [0,T].$$

Now

$$|y(t)|^p \le [k_1(t)]^p \left(\int_0^T k_2(s)v(s)ds \right)^p \quad \text{for a.e. } t \in [0,T]$$

so

$$\|y\|_p \le \|k_1\|_p \int_0^T k_2(s)v(s)ds. \tag{2.12}$$

Combining this with (2.6) gives

$$y(t) \ge M \int_0^T k_1(t)k_2(s)v(s)ds \ge M \, \frac{k_1(t)}{\|k_1\|_p} \, \|y\|_p = a(t) \, \|y\|_p \quad \text{for a.e. } t \in [0,T].$$

Thus $y \in C$ so $A\colon C \to 2^C$. Also notice [3,6] guarantees that

$$A\colon C \to 2^C \text{ is upper semicontinuous.} \tag{2.13}$$

In addition note [8,9,10:pp. 47–49] implies $K\colon L^{p_2}[0,T] \to L^p[0,T]$ is completely continuous, and $N_f\colon L^p[0,T] \to 2^{L^{p_2}[0,T]}$ maps bounded sets into bounded sets. Consequently

$$A\colon C \to K(C) \text{ is completely continuous.} \tag{2.14}$$

Let

$$\Omega_\alpha = \{y \in L^p[0,T] : \|y\|_p < \alpha\} \quad \text{and} \quad \Omega_\beta = \{y \in L^p[0,T] : \|y\|_p < \beta\}.$$

Assume that $\beta < \alpha$ (a similar argument holds if $\alpha < \beta$). It is immediate from (2.13) and (2.14) that

$$A\colon C \cap \overline{\Omega_\alpha} \to K(C) \text{ is upper semicontinuous and compact.}$$

If we show

$$\|y\|_p < \|x\|_p \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\alpha \tag{2.15}$$

and

$$\|y\|_p > \|x\|_p \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\beta \tag{2.16}$$

are true, then Theorem 1.1 guarantees that the operator A has a fixed point in $C \cap (\overline{\Omega_\alpha} \backslash \Omega_\beta)$. This in turn implies that (2.1) has at least one solution $y \in L^p[0,T]$ with $\beta \le \|y\|_p \le \alpha$ and $y(t) \ge a(t)\beta$ for a.e. $t \in [0,T]$.

Suppose $x \in C \cap \partial\Omega_\alpha$, so $\|x\|_p = \alpha$, and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t \in [0,T].$$

Now (2.4) and (2.6) guarantee that

$$|y(t)| \leq k_1(t) \int_0^T k_2(s)\left[|a_1(s)| + a_2|x(s)|^{\frac{p}{p_2}}\right] ds \quad \text{for a.e. } t \in [0,T].$$

This together with (2.9) yields

$$\begin{aligned}
\|y\|_p &\leq \|k_1\|_p \|k_2\|_{p_1} \left(\int_0^T \left[|a_1(s)| + a_2 \, |x(s)|^{\frac{p}{p_2}} \right]^{p_2} ds \right)^{\frac{1}{p_2}} \\
&\leq \|k_1\|_p \|k_2\|_{p_1} \left(2^{p_2-1} \int_0^T [|a_1(s)|^{p_2} + [a_2]^{p_2} |x(s)|^p] ds \right)^{\frac{1}{p_2}} \\
&= 2^{\frac{p_2-1}{p_2}} \|k_1\|_p \, \|k_2\|_{p_1} \left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2} \, \|x\|_p^p \right)^{\frac{1}{p_2}} \\
&= 2^{\frac{p_2-1}{p_2}} \, \|k_1\|_p \, \|k_2\|_{p_1} \left(\|a_1\|_{p_2}^{p_2} + [a_2]^{p_2} \, \alpha^p \right)^{\frac{1}{p_2}} \\
&< \alpha = \|x\|_p
\end{aligned}$$

and so (2.15) is satisfied.

Now suppose $x \in C \cap \partial\Omega_\beta$, so $\|x\|_p = \beta$ and $x(t) \geq a(t)\beta$ for a.e. $t \in [0,T]$, and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for a.e. } t \in [0,T].$$

Notice (2.8) guarantees that $v(s) \geq q(s)\psi(x(s))$ for a.e. $s \in [0,T]$ and this together with (2.6) yields

$$y(t) \geq Mk_1(t) \int_0^T k_2(s)q(s)\psi(x(s))ds \quad \text{for a.e. } t \in [0,T].$$

Combining with (2.10) gives

$$\begin{aligned}
\|y\|_p &\geq M \, \|k_1\|_p \int_0^T k_2(s)q(s)\psi(x(s))ds \\
&\geq M\|k_1\|_p \int_0^T k_2(s)q(s)\psi(a(s)\beta)ds \\
&> \beta = \|x\|_p
\end{aligned}$$

and thus (2.16) is satisfied. Now apply Theorem 1.1. □

3. $C[0,T]$ SOLUTIONS

In this section we discuss the Fredholm integral inclusion

$$y(t) \in \int_0^T k(t,s)f(s,y(s))ds \quad \text{for } t \in [0,T], \tag{3.1}$$

where k: $[0,T] \times [0,T] \to \mathbf{R}$ and f: $[0,T] \times \mathbf{R} \to K(\mathbf{R})$. We will use Theorem 1.1 to establish the existence of a nonnegative solution $y \in C[0,T]$ to (3.1). We will let $|\cdot|_0$ denote the usual norm on $C[0,T]$ i.e., $|u|_0 = \sup_{[0,T]}|u(t)|$ for $u \in C[0,T]$.

Theorem 3.1: *Let $1 \le p < \infty$ and q, $1 < q \le \infty$, the conjugate to p, k: $[0,T] \times [0,T] \to \mathbf{R}$, f: $[0,T] \times \mathbf{R} \to K(\mathbf{R})$ and assume the following conditions are satisfied:*

$$\text{for each } t \in [0,T], \text{ the map } s \mapsto k(t,s) \text{ is measurable;} \tag{3.2}$$

$$\sup_{t\in[0,T]} \left(\int_0^T |k(t,s)|^q \, ds \right)^{\frac{1}{q}} < \infty; \tag{3.3}$$

$$\int_0^T |k(t',s) - k(t,s)|^q \, ds \to 0 \quad \text{as } t \to t', \text{ for each } t' \in [0,T]; \tag{3.4}$$

$$\text{for each } t \in [0,T], \quad k(t,s) \ge 0 \quad \text{for a.e. } s \in [0,T]; \tag{3.5}$$

$$\begin{array}{l}\text{for each measurable } u\text{: } [0,T] \to \mathbf{R}, \text{ the map } t \mapsto f(t,u(t)) \\ \text{has measurable single valued selections;}\end{array} \tag{3.6}$$

$$\text{for a.e. } t \in [0,T], \text{ the map } u \mapsto f(t,u) \text{ is upper semicontinuous;} \tag{3.7}$$

$$\begin{array}{l}\text{for each } r > 0, \exists\, h_r \in L^p[0,T] \text{ with } |f(t,y)| \le h_r(t) \\ \text{for a.e. } t \in [0,T] \text{ and every } y \in \mathbf{R} \text{ with } |y| \le r;\end{array} \tag{3.8}$$

$$\text{for a.e. } t \in [0,T] \text{ and all } y \in (0,\infty), u > 0 \text{ for all } u \in f(t,y); \tag{3.9}$$

$$\begin{array}{l}\exists\, g \in L^q[0,T] \text{ with } g\text{: } [0,T] \to (0,\infty) \text{ and} \\ \text{with } k(t,s) \le g(s) \text{ for } t \in [0,T];\end{array} \tag{3.10}$$

$$\begin{array}{l}\exists\, \delta, \epsilon, 0 \le \delta < \epsilon \le T \text{ and } M, 0 < M < 1, \\ \text{with } k(t,s) \ge M\, g(s) \text{ for } t \in [\delta,\epsilon];\end{array} \tag{3.11}$$

$$\begin{array}{l}\exists\, h \in L^p[0,T] \text{ with } h\text{: } [0,T] \to (0,\infty), \text{ and } w \ge 0 \text{ continuous} \\ \text{and nondecreasing on } (0,\infty) \text{ with } |f(t,y)| \le h(t)\, w(y) \\ \text{for a.e. } t \in [0,T] \text{ and all } y \in (0,\infty);\end{array} \tag{3.12}$$

$$\begin{array}{l}\exists\, \tau \in L^p[\delta,\epsilon] \text{ with } \tau > 0 \text{ a.e. on } [\delta,\epsilon] \text{ and with for a.e.} \\ t \in [\delta,\epsilon] \text{ and } y \in (0,\infty), u \ge \tau(t)\, w(y) \text{ for all } u \in f(t,y);\end{array} \tag{3.13}$$

$$\exists\, \alpha > 0 \quad \text{with} \quad 1 < \frac{\alpha}{w(\alpha)\sup_{t\in[0,T]}\int_0^T k(t,s)h(s)ds} \tag{3.14}$$

and

$$\exists \beta > 0,\ \beta \neq \alpha \quad \text{with} \quad 1 > \frac{\beta}{w(M\beta) \int_\delta^\epsilon \tau(s)\, k(\sigma, s)\, ds}; \tag{3.15}$$

here $\sigma \in [0, T]$ *is such that*

$$\int_\delta^\epsilon \tau(s) k(\sigma, s) ds = \sup_{t \in [0,T]} \int_\delta^\epsilon \tau(s) k(t, s) ds. \tag{3.16}$$

Then (3.1) has at least one nonnegative solution $y \in C[0, T]$ *and either*

(A) $0 < \alpha < |y|_0 < \beta$ *and* $y(t) \geq M\alpha$ *for* $t \in [\delta, \epsilon]$ *if* $\alpha < \beta$

or

(B) $0 < \beta < |y|_0 < \alpha$ *and* $y(t) \geq M\beta$ *for* $t \in [\delta, \epsilon]$ *if* $\beta < \alpha$

holds.

Proof: Let $E = (C[0, T], |\cdot|_0)$ and

$$C = \left\{ y \in C[0, T] : y(t) \geq 0 \text{ for } t \in [0, T] \text{ and } \min_{t \in [\delta, \epsilon]} y(t) \geq M|y|_0 \right\}.$$

Also let $A = K \circ N_f$: $C \to 2^E$, where K: $L^p[0, T] \to C[0, T]$ and N_f: $C[0, T] \to 2^{L^p[0,T]}$ are given by

$$Ky(t) = \int_0^T k(t, s) y(s) ds$$

and

$$N_f u = \{y \in L^p[0, T] : y(t) \in f(t, u(t)) \quad \text{a.e. } t \in [0, T]\}.$$

Remark 3.1: Note A is well defined since if $x \in C$ then [4,5] guarantee that $N_f x \neq \emptyset$.

We first show A: $C \to 2^C$. To see this let $x \in C$ and $y \in Ax$. Then there exists a $v \in N_f x$ with

$$y(t) = \int_0^T k(t, s) v(s) ds \quad \text{for } t \in [0, T].$$

This together with (3.10) yields

$$|y(t)| \leq \int_0^T g(s) v(s) ds \quad \text{for } t \in [0, T]$$

and so

$$|y|_0 \leq \int_0^T g(s) v(s) ds. \tag{3.17}$$

On the other hand (3.11) and (3.17) yields

$$\min_{t \in [\delta, \epsilon]} y(t) = \min_{t \in [\delta, \epsilon]} \int_0^T k(t, s) v(s) ds \geq M \int_0^T g(s) v(s) ds \geq M|y|_0,$$

so $y \in C$. Thus $A: C \to 2^C$. A standard result from the literature [5,7,8,10] guarantees that

$$A: C \to K(C) \text{ is upper semicontinuous and completely continuous.}$$

Let

$$\Omega_\alpha = \{u \in C[0,T] : |u|_0 < \alpha\} \quad \text{and} \quad \Omega_\beta = \{u \in C[0,T] : |u|_0 < \beta\}.$$

Without loss of generality assume $\beta < \alpha$. If we show

$$|y|_0 < |x|_0 \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\alpha \tag{3.18}$$

and

$$|y|_0 > |x|_0 \quad \text{for all } y \in Ax \text{ and } x \in C \cap \partial\Omega_\beta \tag{3.19}$$

are true, then Theorem 1.1 guarantees the result.

Suppose $x \in C \cap \partial\Omega_\alpha$, so $|x|_0 = \alpha$, and $y \in Ax$. Then there exists $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)\,ds \quad \text{for } t \in [0,T].$$

Now (3.12) implies that for $t \in [0,T]$ we have

$$\begin{aligned} |y(t)| &\le \int_0^T k(t,s)h(s)w(x(s))\,ds \le w(|x|_0)\int_0^T k(t,s)h(s)\,ds \\ &\le w(\alpha) \sup_{t\in[0,T]} \int_0^T k(t,s)h(s)\,ds. \end{aligned}$$

This together with (3.14) yields

$$|y|_0 \le w(\alpha) \sup_{t\in[0,T]} \int_0^T k(t,s)h(s)ds < \alpha = |x|_0,$$

so (3.18) holds.

Next suppose $x \in C \cap \partial\Omega_\beta$, so $|x|_0 = \beta$ and $M\beta \le x(t) \le \beta$ for $t \in [\delta, \epsilon]$, and $y \in Ax$. Then there exists $v \in N_f x$ with

$$y(t) = \int_0^T k(t,s)v(s)ds \quad \text{for } t \in [0,T].$$

Notice (3.13) and (3.15) imply

$$\begin{aligned} y(\sigma) &= \int_0^T k(\sigma,s)v(s)ds \ge \int_\delta^\epsilon k(\sigma,s)v(s)ds \\ &\ge \int_\delta^\epsilon k(\sigma,s)\tau(s)w(x(s))ds \ge w(M\beta)\int_\delta^\epsilon k(\sigma,s)\tau(s)ds \\ &> \beta = |x|_0. \end{aligned}$$

Thus $|y|_0 > |x|_0$, so (3.19) holds. Now apply Theorem 1.1. □

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