THEORY OF BEAMS

The Application of the Laplace Transformation Method to Engineering Problems

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PREFACE

THIS monograph is addressed mainly to technically minded readers. Its principal aim is to discuss the solutions of the ordinary differential equations with assigned boundary conditions which occur in the theory of beams by means of integral transforms, now called the "Laplace integrals" or "Laplace transformation".

This method of solving linear differential equations was originally published by Laplace nearly one hundred and fifty years ago (1812), and has been used by mathematicians alongside the other methods as a regular training in higher mathematics for a good many years. Engineers, on the other hand, whose mathematical training, of necessity, covers a more restricted range are still not very familiar with the method. But following the natural evolution of events, the engineer finds nowadays that he is continuously requiring more and more mathematical knowledge, i.e. more powerful mathematical tools to be able to solve the more complicated problems in ever expanding fields of engineering science. And so the engineer came in recent years to realize the usefulness of the Laplace transformation method.

The method provides a form of shorthand by means of which very quick and elegant solutions of a great many engineering problems may be obtained, often saving much of the laborious calculations involved in the classical approach. It may be objected that these solutions are in a rather symbolic form and in fact require conversion into the more practical form in order to be directly amenable to physical interpretation and this, in turn, may be either a difficult or laborious process. However, the solutions can be given in standardized forms, e.g. arranged as tables similar, say, to those of logarithms or other specific functions, thus greatly simplifying the actual procedure of analysis. And once the technique has been mastered, the method of solution becomes more direct and straightforward than the "classical" methods.

E. P. B.

Sheffield, September 1957.

SUMMARY

This monograph is the first part of a major project on the application of the Laplace transformation method to static problems in theory of structures and deals with beams on two and more supports under various conditions of loading.

Beams of constant rigidity or composed of sections of constant rigidity are discussed first. Starting with the differential equation of the deflexion curve and with a uniformly distributed loading, the solution of the equation is obtained by means of the Laplace transformation. Then, by means of a suitable definition of concentrated forces and couples, the author generalizes the solution to include the case of simultaneous loading by concentrated forces and couples as well as distributed loads (uniform and non-uniform).

It is shown that the elastic curve of a beam loaded in such a general manner and resting on two or more supports is given by the relation $y = W(x) + S(x)$, where $W(x)$ and $S(x)$ are *step functions* defined and briefly discussed in the introduction. The function $W(x)$ is a polynomial whose coefficients depend entirely upon the nature of the supports and can be evaluated from the boundary conditions, while the second function $S(x)$, called by the author the *load function*, depends solely upon the loading of the beam. If the load distribution is known, the load function can be determined beforehand independently of the step polynomial $W(x)$ either by direct application of the Laplace transformation or by means of tables of the Laplace transforms.

In this way the problem of finding the deflexion curve is transformed from the problem of solving a differential equation to that of solving algebraic equations. In short, it reduces to finding the coefficients of the polynomial $W(x)$ by solving a system of simultaneous algebraic equations. Moreover, the use of this method does away with the necessity of different treatment of the iso- and hyper-static beams since the same general relation holds in both cases.

Solutions are obtained for beams on two support under various conditions of end fixing (freely supported, built-in, etc.) as well as for continuous beams both with rigid and elastic supports. Finally, the author derives a solution of the differential equation of the elastic curve for the case of a beam with variable rigidity and defines the load function for this case.

I. INTRODUCTORY INFORMATION

§ 1. Step functions and multi-step functions

Suppose we are given two functions : $f_1(x)$ which is defined and continuous in the open region (a, c), and $f_2(x)$ which is defined and continuous in the open region (c, b), where $a < c < b$. Assuming that these functions are finite at both ends of their respective regions, a new function over the whole region (a, b) can now be defined by means of them as follows

$$f(x) = \begin{cases} f_1(x), & a < x < c, \\ f_2(x), & c < x < b. \end{cases} \qquad \ldots .(1.1.1)$$

Alternatively eq. (1.1.1) may be written in a more convenient form for our purposes as

$$f(x) = \langle f_1(x) \rangle_a^c + \langle f_2(x) \rangle_c^b . \qquad \ldots .(1.1.2)$$

The function $f(x)$ so defined will be called the *step function*, and the terms such as $\langle f_1(x)\rangle_a^c$ and $\langle f_2(x)\rangle_c^b$ will be referred to as its *elements*. Such a function is still to be defined at the three points (a, b, c), corresponding to the ends of the intervals (a, c) and (c, b). This can be done in several ways. However, a convenient method is to make use of the left and right end limits of the elements, i.e. $f_1(a+0)$, $f_1(c-0)$ and $f_2(c+0)$, $f_2(b-0)$, and to ascribe to the function $f(x)$ at the ends of the region (a, b) the outer limits $f_1(a+0)$ and $f_2(b-0)$, while at the intermediate point c to treat $f(x)$ as double-valued, i. e. either $f_1(c-0)$ or $f_2(c+0)$ depending whether the interval (a, c) or (c, b) is considered.

The need for such a double-valued step function at certain points arises, for example, when considering the distribution of the shear force in a transversely loaded beam. At the points of application of the concentrated loads or at the supports, the shearing force is double-valued.

The graph of the step function

$$f(x) = \langle y_1 \rangle_a^c + \langle y_2 \rangle_c^b, \qquad \ldots.(1.1.3)$$

where y_1 and y_2 are constant, is shown in Fig. 1.

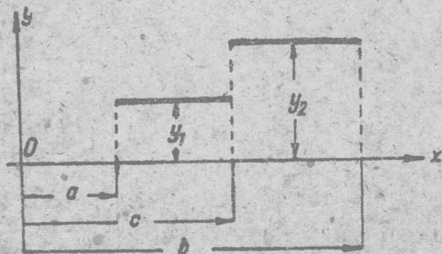

FIG. 1

From the above discussion it is not to be assumed that a step function must have points where it is double-valued; it is possible that $f_1(c-0) = f_2(c+0)$. In such a case, in accordance with the definition, this common limit will be the value of the step function $f(x)$ at c, and the function is single-valued over the whole region.

As an example of such a single-valued step function we have

$$f(x) = \langle kx \rangle_0^c + \langle kx + P(x-c)^3 \rangle_c^b \quad (0 < c < b). \qquad \ldots (1.1.4)$$

Its graph is given in Fig. 2.

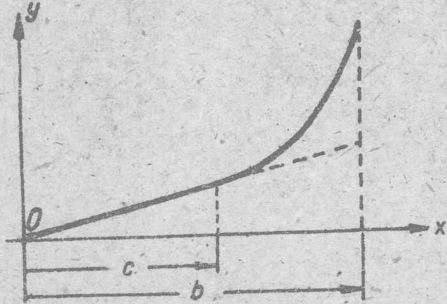

Fig. 2

If there are more than two elements, such a function will be called *multi-step* function and then we can write

$$f(x) = \sum_{i=0}^{n-1} \langle f_i(x) \rangle_{a_i}^{a_{i+1}} \quad (a = a_0 < a_1 < \cdots < a_n = b). \qquad \ldots (1.1.5)$$

It will be found advantageous to retain the above notation even for such cases where one of the intervals reduces to a point. We may write then

$$\langle f_i(x) \rangle_c^c = f_i(c). \qquad \ldots (1.1.6)$$

The symbol $\langle f_i(x) \rangle_{a_i}^{b_i}$, where $a_i < b_i$, indicates that the function $f_i(x)$ is to be considered only within the interval (a_i, b_i).

Keeping this in mind, we may modify slightly the definition of a step function given in eq. (1.1.2) as follows

$$f(x) = \langle f_1(x) \rangle_a^b + \langle f_2(x) \rangle_c^b \quad (a < c < b). \quad \ldots.(1.1.7)$$

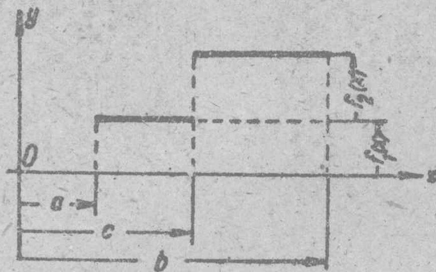

FIG. 3

Its graph is given in Fig. 3. It is easily seen that the elements of the step function (1.1.7) overlap. Eq. (1.1.7), of course, reduces to eq. (1.1.2), viz.:

$$\langle f_1 \rangle_a^b + \langle f_2 \rangle_c^b = \langle f_1 \rangle_a^c + \langle f_1 \rangle_c^b + \langle f_2 \rangle_c^b =$$
$$= \langle f_1 \rangle_a^c + \langle f_1 + f_2 \rangle_c^b. \quad \ldots.(1.1.8)$$

When a step function, given in the form (1.1.7), has an element extending over the whole region under consideration (a, b), we shall leave out the sign $\langle\ \rangle_a^b$ writing

$$f(x) = f_1(x) + \langle f_2(x) \rangle_c^b \quad (a < c < b). \quad \ldots.(1.1.9)$$

Further, if in a step function there is a zero-valued element over an interval, e.g. $\langle 0 \rangle_c^d$, we shall omit it all together ; and conversely, if in some interval in the region (a, b) the step function is not given, it is to be understood that its value is zero in that interval. Hence every sum of the form

$$\sum_{i=1}^{n} \langle f_i \rangle_{a_i}^{b_i} \qquad \ldots .(1.1.10)$$

denotes a step function in the region (a, b) if only $a \leq a_i < b$ or $a < b_i \leq b$ or if $a_i \leq b_i$, and the functions f_i are determined respectively in the intervals $(a_i\, b_i)$ and have finite limits $f_i\,(a_i+0)$ and $f_i\,(b_i-0)$.

Here are some examples on the application of step functions.

1. If a beam of length l, with its neutral axis along the Ox-axis, has a flexural rigidity B_1 in the first half of its length and B_2 in the second half, then its rigidity over the whole length can be expressed as

$$B\,(x) = \langle B_1 \rangle_0^{l/2} + \langle B_2 \rangle_{l/2}^{l}. \qquad \ldots .(1.1.11)$$

2. If the beam given above is acted on by a uniformly distributed load q_1 in the interval $(0, \frac{1}{2}\, l)$ and in addition there is another distributed load q_2 over the interval $(\frac{1}{4}\, l, \frac{3}{4}\, l)$ while the remainder of the beam $(\frac{3}{4}\, l, l)$ is unloaded, the load over the whole length may be expressed as follows

$$q\,(x) = \langle q_1 \rangle_0^{l/2} + \langle q_2 \rangle_{l/4}^{3l/4} \qquad \ldots .(1.1.12)$$

or

$$q\,(x) = \langle q_1 \rangle_0^{l/4} + \langle q_1 + q_2 \rangle_{l/4}^{l/2} + \langle q_2 \rangle_{l/2}^{3l/4}. \qquad \ldots .(1.1.12.1)$$

or alternatively

$$q\,(x) = \langle q_1 \rangle_0^{l/4} + \langle q_1 + q_2 \rangle_{l/4}^{l/2} + \langle q_2 \rangle_{l/2}^{3l/4} + \langle 0 \rangle_{3l/4}^{l}. \qquad \ldots .(1.1.12.2)$$

The graphs of these functions are given in Fig. 4.

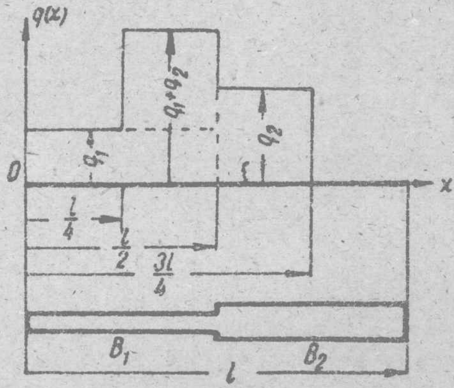

FIG. 4

3. The reciprocal of the rigidity of the above beam is given by

$$\frac{1}{B(x)} = \left\langle \frac{1}{B_1} \right\rangle_0^{l/2} + \left\langle \frac{1}{B_2} \right\rangle_{l/2}^{l} \quad \ldots .(1.1.13)$$

or

$$\frac{1}{B(x)} = \left\langle \frac{1}{B_1} \right\rangle_0^{l/4} + \left\langle \frac{1}{B_1} \right\rangle_{l/4}^{l/2} + \left\langle \frac{1}{B_2} \right\rangle_{l/2}^{3l/4} + \left\langle \frac{1}{B_2} \right\rangle_{3l/4}^{l} . \quad \ldots .(1.1.13.1)$$

Using relations (1.1.12.2) and (1.1.13.1) the differential equation of the deflected beam

$$y^{IV} = \frac{q(x)}{B} \qquad (B = \text{const.}) \quad \ldots .(1.1.14)$$

can be written as follows

$$y^{IV} = \left\langle \frac{q_1}{B_1} \right\rangle_0^{l/4} + \left\langle \frac{q_1 + q_2}{B_1} \right\rangle_{l/4}^{l/2} + \left\langle \frac{q_2}{B_2} \right\rangle_{l/2}^{3l/4} + \langle 0 \rangle_{3l/4}^{l} . \quad \ldots .(1.1.14.1)$$