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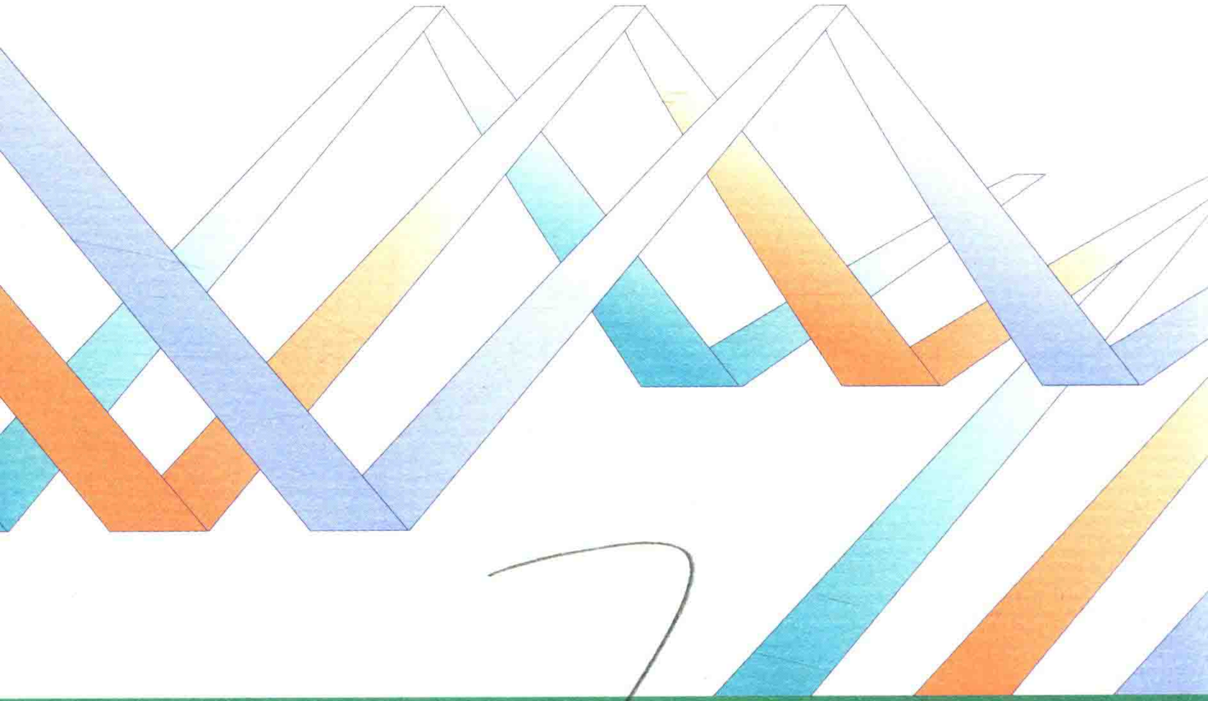

Introduction to Moduli Spaces of Riemann Surfaces and Tropical Curves

黎曼曲面和热带曲线的模空间导引

Lizhen Ji · Eduard Looijenga

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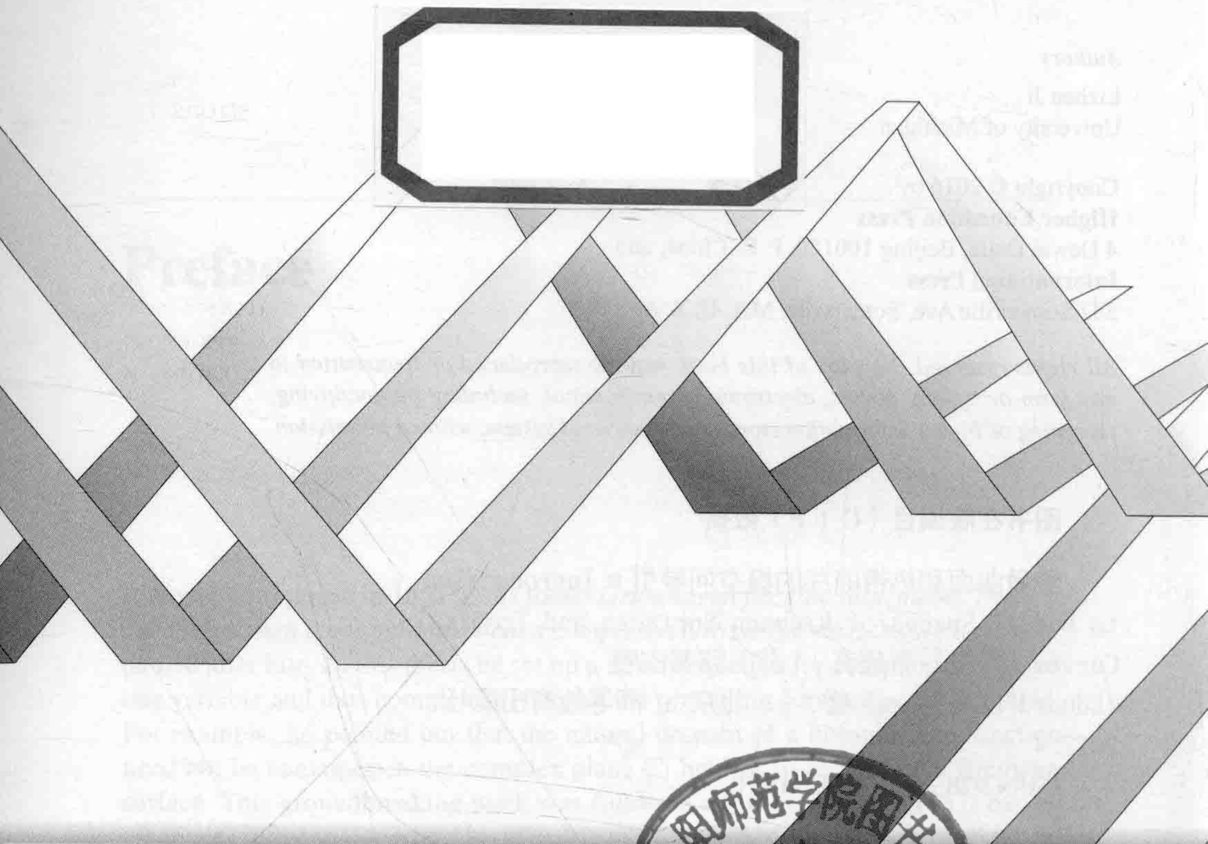

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Lizhen Ji · Eduard Looijenga

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Surveys of Modern Mathematics

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Mathematics has developed to a very high level and is still developing rapidly. An important feature of the modern mathematics is strong interaction between different areas of mathematics. It is both fruitful and beautiful. For further development in mathematics, it is crucial to educate students and younger generations of mathematicians about important theories and recent developments in mathematics. For this purpose, accessible books that instruct and inform the reader are crucial. This new book series "Surveys of Modern Mathematics" (SMM) is especially created with this purpose in mind. Books in SMM will consist of either lecture notes of introductory courses, collections of survey papers, expository monographs on well-known or developing topics.

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Preface

Riemann introduced in 1851 in his thesis *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* the surfaces we have now named after him. In this thesis he set up a new foundation for complex analysis of one variable and thus completely changed the prevailing perspective on that field. For example, he pointed out that the natural domain of a holomorphic function need not be contained in the complex plane $\mathbb{C}$, but lies in general on a Riemann surface. This groundbreaking work was followed a few years later (1857) by another very influential paper of his, *Theorie der Abel'schen Functionen*, in which he developed systematically the topology of Riemann surfaces, discussed integrals of holomorphic and meromorphic 1-forms and proved the crucial Riemann inequality in the Riemann-Roch theorem, thereby establishing the equivalence between compact Riemann surfaces and plane algebraic curves over the complex number field. Moreover, in order to classify compact Riemann surfaces or rather function fields of algebraic curves, he introduced the notion of "moduli" for such surfaces, and for the purpose of solving the Jacobi inversion problem for a general compact Riemann surface, he developed the theory of (what we now call) Riemann theta functions. Although analysis played a crucial role, this paper also initiated the algebraic geometry (in particular the birational geometry) of algebraic curves. So Riemann was fully aware of the rich world of connections between his surfaces and algebraic curves.

Much has happened since, and both Riemann surfaces and their moduli spaces have come to assume a central role in several areas in mathematics. The moduli spaces in question are considered to be among the most important algebraic varieties, but have also a prominent place in string theory. The subject is still very much vibrant as the focus of current research, with many of its central questions still being unanswered.

A milestone in the study of moduli spaces of Riemann surfaces was the introduction of Teichmüller spaces endowed with the action of a mapping class group, with the orbit space then yielding the moduli space. This view point provided an analogy with locally symmetric spaces, which by definition are obtained as orbit space of symmetric spaces endowed with an action of an arithmetic group. This analogy, though imperfect, has been very fruitful in the past decades, as it suggested both problems and solutions.

As we mentioned, compact Riemann surfaces can also be regarded as algebraic curves over the complex numbers. In the past few years, there has been an explosion of work on their tropical counterparts, i.e., algebraic curves over the tropical semifield. These have appeared in many unexpected topics and make currently a very active area of study. It turns out that the moduli spaces of tropical curves also fit into the above analogy, in the sense that these are the orbit spaces of tropical Teichmüller spaces by outer automorphism groups of free groups. The outer automorphism groups of free groups are among the most important groups in geometric group theory, and the tropical Teichmüller spaces were studied as spaces of marked metric graphs before the subject of tropical geometry (and its name) existed.

Given the rich history of Riemann surfaces and their moduli spaces, and their generalizations, applications and connections with other subjects, it seems helpful to provide an accessible and timely introduction to all the above topics, while emphasizing the underlying connections. The current book is an attempt towards this goal. It consists of two parts. The first part deals with mapping class groups of surfaces, Teichmüller spaces and their applications to moduli spaces of Riemann surfaces, whereas the second part deals with tropical analogues and some applications in geometric group theory. Though these parts were conceived independently, together they cover both basic and essential results in these subjects as well as some recent developments. Although there exist several works on some of the above topics, we hope and believe that the nature of the subject justifies our offering of our own perspective on this wonderful world.

Lizhen Ji
Eduard Looijenga

September 2016

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Part I

Moduli Spaces of Riemann Surfaces

by Eduard Looijenga

This is the write up of a course I taught at Tsinghua University in the Fall of 2011, while being supported by the there-based Mathematics Sciences Center.

One can enter the subject via algebraic geometry, complex analysis, (combinatorial) topology and even homotopy theory. My aim was not to confine myself to just one of these approaches, but rather to give the students a sense how they not only supplement, but sometimes also reinforce each other. That sounds rather ambitious and so I hasten to add that the course's content was not only subject to time constraints, but also to my probable bias and the limitations of my knowledge. Having said that, I believe that the implementation of this philosophy not only exhibits the intertwined nature mentioned above, but also allows occasionally for shorter proofs, at least shorter than the ones I found in the literature.

In the version prepared during the course, references were absent. This has now been remedied, to the extent that I provided them where I believed the text demanded this. Of course, this does not do the literature justice and the reader is in this regard probably best served by the references list in the recent book by Arbarello-Cornalba-Griffiths [1]. This magnum opus of almost a thousand pages is also an excellent reference for much of the material that is discussed here and I highly recommend to use it on the side and for further study.

I knew from the outset that this 'multidisciplinary approach' would be rather demanding on the students. But my audience, which even counted a few undergraduates, proved to be very motivated and gave me plenty of feedback (which occasionally led to a correction or a more detailed discussion). The course was a joy to give.

In these notes a *surface* always means an oriented 2-manifold admitting a countable base. We denote the unit circle if $\mathbb{C}$ of complex numbers of norm 1 by $\mathbb{C}_1$.

1 Mapping class groups and Dehn twists

Mapping class groups

Let S be a surface. For any subset $P \subset S$. We denote by $\mathrm{Diff}(S,P)$ the group of diffeomorphisms of S onto S that are the identity on P (but usually write $\mathrm{Diff}(S)$ when $P = \emptyset$). We write $\mathrm{Diff}^\circ(S,P) \subset \mathrm{Diff}(S,P)$ for its identity component—this is a normal subgroup. Two elements $h, h' \in \mathrm{Diff}(S)$ are called *isotopic relative to* P if they lie in the same path component of $\mathrm{Diff}(S,P)$, in other words, if $h^{-1}h'$ in $\mathrm{Diff}^\circ(S,P)$.

The group of orientation preserving diffeomorphisms in $\mathrm{Diff}(S,P)$, denoted by $\mathrm{Diff}^+(S,P) \subset \mathrm{Diff}(S,P)$, contains $\mathrm{Diff}^\circ(S,P)$ as a normal subgroup. We call the quotient $\mathrm{Mod}(S,P) := \mathrm{Diff}^+(S,P)/\mathrm{Diff}^\circ(S,P)$ (which comes endowed with the discrete topology) the *mapping class group* of (S,P). As these notions only depend on the closure of P in S, we may as well assume that P is already closed in S.

Dehn twists

We now assume $P \subset S$ closed. The collection of embedded circles in $S \smallsetminus P$ is acted on by $\mathrm{Diff}(S,P)$. Two embedded circles α, α' are said to be isotopic relative to P if they have the same $\mathrm{Diff}^\circ(S,P)$-orbit. So $\mathrm{Mod}(S,P)$ acts on the set of isotopy classes of embedded circles in $S \smallsetminus P$.

An embedded circle $\alpha \subset S \smallsetminus P$ leads to a so-called Dehn twist $D_\alpha \in \mathrm{Mod}(S,P)$: let $\phi : (-1,1) \times \mathbb{C}_1 \to S \smallsetminus P$ be an open, *orientation preserving* embedding such that α is the image of ϕ_0 and let $f : (-1,1) \to [0,2\pi]$ be a smooth function which is constant 0 on $(-1,-\frac{1}{2})$ and constant 2π on $(\frac{1}{2},1)$. Let $h : S \to S$ be the identity outside the image of ϕ and be such that $h(\phi(t,u)) = \phi(t, ue^{-\sqrt{-1}f(t)})$. Then h is a diffeomorphism and one can check that its image in $\mathrm{Mod}(S,P)$ only depends on the isotopy class relative to P of α. In particular, it does not depend on an orientation of α. We call this element of $\mathrm{Mod}(S,P)$ the *Dehn twist* associated to α and denote it by D_α.

Exercise 1.1. Prove that D_α is trivial when α bounds a disk in S which meets P in at most one point.

We will be mostly interested in the case when S is a closed surface and $P \subset S$ is finite subset. We then call the pair (S,P) a *P-pointed surface*. We then often write S° for $S \smallsetminus P$. If S is connected and g is the genus of S, then we sometimes denote $\mathrm{Mod}(S,P)$ by $\mathrm{Mod}_{g,P}$, or if the elements of P have been effectively numbered: $P = \{p_1, \dots, p_n\}$, by $\mathrm{Mod}_{g,n}$.

It can be shown that $\mathrm{Mod}_{g,P}$ is finitely presented with Dehn twists as generators. We shall see that $\mathrm{Mod}_{0,3}$ is trivial and that $\mathrm{Mod}_{1,1} \cong \mathrm{SL}(2,\mathbb{Z})$.

Now assume that S is a closed connected surface of genus g and that P is finite. The $\mathrm{Diff}^+(S,P)$-orbits of embedded circles in S° can then be topologically distinguished: a *nonseparating* embedded circle $\alpha \subset S^\circ$ is by definition such that its complement $S \setminus \alpha$ connected. That complement is then the interior of a compact connected surface of genus $g-1$ with two boundary components. Otherwise α is *separating* and splits S into two connected components S', S'', each of which has as its closure a compact subsurface with α as boundary. If the genera of these surfaces are denoted by g' and g'', then $g = g' + g''$. The members of P will divide themselves in $P' := P \cap S'$ and $P'' := P \cap S''$. The diffeomorphism type of a compact surface with boundary is completely given by its genus and its number of boundary components and this is still true in the relative situation where the surface has been equipped with a finite subset in its interior. From this we easily deduce:

Proposition 1.2. *The* non-separating *embedded circles in* S° *make up a single orbit under* $\mathrm{Diff}^+(S,P)$ *and for a* separating *embedded circle its* $\mathrm{Diff}^+(S,P)$*-orbit is characterized by the unordered pair of pairs* $\{(g',P'),(g'',P'')\}$ *defined above. This gives rise to a corresponding characterization of the* $\mathrm{Mod}(S,P)$*-conjugacy classes of Dehn twists.*

The notion of a mapping class group does not change if we pass to the topological setting: any closed topological surface admits a differentiable structure, and for (S,P) as above the inclusion $\mathrm{Diff}^+(S,P) \subset \mathrm{Homeo}^+(S,P)$ induces an isomorphism on their path component groups $\mathrm{Mod}(S,P) \xrightarrow{\cong} \mathrm{Homeo}^+(S,P)/\mathrm{Homeo}^\circ(S,P)$. This is even true if we descend to the homotopy category: the natural map from $\mathrm{Mod}(S,P)$ to the group $\mathrm{Htp}(S,P)$ of homotopy equivalences $(S,P) \to (S,P)$ relative to P is an isomorphism. That property amounts to a characterization of $\mathrm{Mod}(S,P)$ as a group of *outer automorphisms* of the fundamental group of S°, which we describe the next subsection. Before we do so, we mention that there is an intermediate structure which is quite useful in Teichmüller theory, namely that of a quasi-conformal structure. We will not give the definition, but just mention that the connected component group of the group of automorphisms of that structure which preserve orientation and fix P pointwise also maps isomorphically to $\mathrm{Mod}(S,P)$.

Fundamental groups and mapping class groups

We will here describe without proof a characterization of a mapping class group in terms of its (outer) action on the fundamental group. We begin with general discussion of outer actions of groups.

Recall that given a group π, the group $\mathrm{Aut}(\pi)$ of group automorphisms of π contains the group $\mathrm{Inn}(\pi)$ of its inner automorphisms as a normal subgroup. The quotient group $\mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ is called the group of its *outer automorphisms* and is denoted $\mathrm{Out}(\pi)$. Since $\mathrm{Inn}(\pi)$ acts trivially on the (co)homology of π, the group $\mathrm{Aut}(\pi)$ acts on this (co)homology always via $\mathrm{Out}(\pi)$. If $B \subset \pi$ is a $\mathrm{Inn}(\pi)$-invariant subset, in other words, a union conjugacy classes of π, then we can form the subgroup $\mathrm{Aut}(\pi, B) \subset \mathrm{Aut}(\pi)$ of automorphisms which preserve each conjugacy class in B and as this this contains $\mathrm{Inn}(\pi)$, we also have defined the quotient group $\mathrm{Out}(\pi, B) := \mathrm{Aut}(\pi, B)/\mathrm{Inn}(\pi)$.

Let X be a path connected space. Its *fundamental groupoid* π_X is the category whose objects are the points of X and for which a morphism from $p \in X$ to $q \in X$ is a homotopy class of arcs from p to q. It is a groupoid since every morphism is an isomorphism. The fundamental group based at $p \in X$, $\pi(X,p)$, then appears as the group of π_X-endomorphisms of p (this presupposes that elements of $\pi(X,p)$ are read from right to left). Given $p, q \in X$, then by assumption there exists a π_X-morphism $\gamma : \pi(X,p) \cong \pi(X,q)$ and any two such differ by an element of $\pi(X,q)$. So the resulting isomorphism $\mathrm{Out}(\gamma) : \mathrm{Out}(\pi(X,p)) \cong \mathrm{Out}(\pi(X,q))$ is independent of the choice of γ. We denote by $\mathrm{Out}(\pi_X)$ this common group. Or if we insist on treating all the points of X on an equal footing: an element of $\mathrm{Out}(\pi_X)$ is the subgroup of $\prod_{p \in X} \mathrm{Out}(\pi(X,p))$ of elements whose components are related by the isomorphisms $\mathrm{Out}(\gamma)$. This renders evident the observation that a homotopy equivalence $h : X \to X$ induces an element of $\pi(h) \in \mathrm{Out}(\pi_X)$ and that thus is defined a group homomorphism from the group $\mathrm{Htp}(X)$ of homotopy classes of self homotopy equivalences $X \to X$ to $\mathrm{Out}(\pi_X)$, to which one often refers as an *outer action* (of $\mathrm{Htp}(X)$) on the fundamental group.

Let us return to S, a closed connected oriented surface. If $o \in S$ is a base point, then it is well-known that $\pi(S,o)$ has a (standard) presentation with generators $\alpha_{\pm 1}, \dots, \alpha_{\pm g}$, and relation $[\alpha_g, \alpha_{-g}] \cdots [\alpha_1, \alpha_{-1}] \equiv 1$.

The outer action of $\mathrm{Diff}^+(S)$ on $\pi_S = \pi(S,o)$ defines a group homomorphism

$$\mathrm{Mod}(S) \to \mathrm{Out}(\pi_S).$$

The theorem alluded to above asserts that for genus $g > 0$ this is an isomorphism onto a subgroup $\mathrm{Out}^+(\pi_S)$ of $\mathrm{Out}(\pi_S)$ of index 2. (The preservation of orientation can be expressed in terms of π_S, because $H^2(S)$ can be understood as group cohomology: $H^2(S) = H^2(\pi_S)$; we give another description below which avoids this language.) For $g = 0$, $\mathrm{Mod}(S)$ is trivial. The above result implies among other things that $\mathrm{Mod}(S)$ is finitely generated and even finitely presented. We will later give another proof of this fact (Corollary 7.3).

For a cofinite subgroup of π_S (i.e., a subgroup of finite index), the collection of its $\mathrm{Aut}(\pi_S)$-conjugates is finite in number (for π_S is finitely generated) so that the intersection of these is still cofinite in π_S. It is clear that for such an $\mathrm{Aut}(\pi_S)$-invariant cofinite subgroup π of π_S is normal in π_S and that we have an action of $\mathrm{Aut}(\pi_S)$ on the finite group π_S/π. The kernel of this action produces a cofinite subgroup of $\mathrm{Aut}(\pi_S)$ and its intersection with $\mathrm{Mod}(S)$ will be cofinite in the latter. A subgroup of $\mathrm{Mod}(S)$ which contains the kernel of such a homomorphism is called a *congruence subgroup* of $\mathrm{Mod}(S)$. Such a subgroup is clearly cofinite in $\mathrm{Mod}(S)$. It is still an open question whether the converse holds:

Question 1.3 (Grothendieck)**.** Is every cofinite subgroup of $\mathrm{Mod}(S)$ a congruence subgroup?

We assume $P \subset S$ finite nonempty and write $S^\circ := S \smallsetminus P$ as before. We also assume that when $g = 0$, $|P| \geq 2$ (we will see that $\mathrm{Mod}(S,P)$ is trivial otherwise). Choose $o \in S^\circ$. In order to give a presentation of $\pi_{S^\circ} = \pi(S^\circ, o)$, we first number the points of P so that $P := \{p_1, \dots, p_n\}$. Then we choose for $k = 1, \dots, n$, $\beta_k \in \pi(S^\circ, o)$ representing a simple positive loop in S° around p_k based at o such that $\beta_n \beta_{n-1} \cdots \beta_1$ is represented by a positive loop which encircles P and bounds a disk in S. A presentation π_{S° has now generators $\alpha_{\pm 1}, \dots, \alpha_{\pm g}, \beta_1, \dots, \beta_n$ that are subject to the relation

$$\beta_n \beta_{n-1} \cdots \beta_1 [\alpha_g, \alpha_{-g}] \cdots [\alpha_1, \alpha_{-1}] \equiv 1.$$

Since this relation allows us to eliminate β_n, this group is in fact freely generated by $\alpha_{\pm 1}, \dots, \alpha_{\pm g}, \beta_1, \dots, \beta_{n-1}$. However it has some additional structure: the conjugacy class $B_k \subset \pi_{S^\circ}$ of β_k is invariantly defined as it consists of *all* the simple positive loops in S° around p_k based at o. If we divide the group $\pi(S^\circ, o)$ out by the normal subgroup generated by B_k, then we get the fundamental group of $S^\circ \cup \{p_k\}$. We put $B := B_1 \cup \cdots \cup B_n$ and observe that we have an evident map

$$\mathrm{Mod}(S,P) \to \mathrm{Out}(\pi_{S^\circ}, B).$$

Another basic result asserts that this map is also an isomorphism. (There is no orientation issue, for an orientation reversing diffeomorphism maps B_1 to B_1^{-1}, which differs from B_1.) Now consider the exact sequence of groups

$$1 \to \mathrm{Inn}(\pi_{S^\circ}) \to \mathrm{Aut}(\pi_{S^\circ}, B) \to \mathrm{Out}(\pi_{S^\circ}, B) \to 1.$$

Since π_{S° has trivial center (it is a free group), we may identify it with its group of inner automorphisms. The middle term may be identified with $\mathrm{Mod}(S, \tilde{P})$, where $\tilde{P} := P \sqcup \{o\}$ and we thus find the *Birman exact sequence*

$$1 \to \pi_{S^\circ} \to \mathrm{Mod}(S, \tilde{P}) \to \mathrm{Mod}(S,P) \to 1.$$

This is in fact also valid when $P = \emptyset$: $\mathrm{Mod}(S)$ can be identified with group of outer automorphisms h of $\pi(S,o)$ that, in terms of the above presentation, can be lifted to an automorphism $\tilde{h}$ of the free group on the generators $\tilde{\alpha}_{\pm 1}, \dots, \tilde{\alpha}_{\pm g}$ which preserves the conjugacy class of $[\tilde{\alpha}_g, \tilde{\alpha}_{-g}] \cdots [\tilde{\alpha}_1, \tilde{\alpha}_{-1}]$.

2 Conformal structures and a rough classification

Conformal structures

Let T be a real vector space of dimension two. A *conformal structure* on T is an inner product on T given up to scalar multiplication. If T is endowed with an orientation, then a conformal structure yields a notion of angle (rotation over an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is defined) and thus turns T into a complex vector space of