

海外优秀数学类教材系列丛书

THOMSON

影印版

Calculus (Fifth Edition)

微积分 (第5版)

(下册)

□ James Stewart



高等教育出版社
Higher Education Press



海外优秀数学类教材系列丛书

影印版

Calculus

(Fifth Edition)

微积分 (第5版)(下册)

James Stewart

McMaster University



高等教育出版社

Higher Education Press

图字 01-2003-6454 号

James Stewart

Calculus: Early Transcendentals, fifth Edition.

ISBN:0-534-39321-7

Copyright©2003 by Brooks/Cole, a division of Thomson Learning

Original language published by Thomson Learning (a division of Thomson Learning Asia Pte Ltd).

All Rights reserved. 本书原版由汤姆森学习出版集团出版。版权所有,盗印必究。

Higher Education Press is authorized by Thomson Learning to publish and distribute exclusively this English language reprint edition. This edition is authorized for sale in the People's Republic of China only (excluding Hong Kong, Macao SAR and Taiwan).

Unauthorized export of this edition is a violation of the Copyright Act. No part of this publication may be reproduced or distributed by any means, or stored in a database or retrieval system, without the prior written permission of the publisher.

本书英文影印版由汤姆森学习出版集团授权高等教育出版社独家出版发行。此版本仅限在中华人民共和国境内(不包括中国香港、澳门特别行政区及中国台湾)销售。未经授权的本书出口将被视为违反版权法的行为。未经出版者预先书面许可,不得以任何方式复制或发行本书的任何部分。

981-254-456-9

图书在版编目(CIP)数据

微积分.下册=Calculus:第5版/(加)史迪沃特
(Stewart,J.)编著.一影印本.一北京:高等教育出版社,2004.7

(海外优秀数学类教材系列丛书)

ISBN 7-04-014004-7

I.微... II.史... III.微积分-高等学校-教材
-英文 IV.0172

中国版本图书馆CIP数据核字(2004)第036110号

出版发行 高等教育出版社
社 址 北京市西城区德外大街4号
邮政编码 100011
总 机 010-82028899

购书热线 010-64054588
免费咨询 800-810-0598
网 址 <http://www.hep.edu.cn>
<http://www.hep.com.cn>

经 销 新华书店北京发行所
印 刷 北京中科印刷有限公司

开 本 889×1194 1/16
印 张 38.75
字 数 810 000

版 次 2004年7月第1版
印 次 2004年7月第1次印刷
定 价 43.80元

本书如有缺页、倒页、脱页等质量问题,请到所购图书销售部门联系调换。

版权所有 侵权必究

出版者的话

在我国已经加入 WTO、经济全球化的今天,为适应当前我国高校各类创新人才培养的需要,大力推进教育部倡导的双语教学,配合教育部实施的“高等学校教学质量与教学改革工程”和“精品课程”建设的需要,高等教育出版社有计划、大规模地开展了海外优秀数学类系列教材的引进工作。

高等教育出版社和 Pearson Education, John Wiley & Sons, McGraw-Hill, Thomson Learning 等国外出版公司进行了广泛接触,经国外出版公司的推荐并在国内专家的协助下,提交引进版权总数 100 余种。收到样书后,我们聘请了国内高校一线教师、专家、学者参与这些原版教材的评介工作,并参考国内相关专业的课程设置为教学实际情况,从中遴选出了这套优秀教材组织出版。

这批教材普遍具有以下特点:(1)基本上是近 3 年出版的,在国际上被广泛使用,在同类教材中具有相当的权威性;(2)高版次,历经多年教学实践检验,内容翔实准确、反映时代要求;(3)各种教学资源配套整齐,为师生提供了极大的便利;(4)插图精美、丰富,图文并茂,与正文相辅相成;(5)语言简练、流畅、可读性强,比较适合非英语国家的学生阅读。

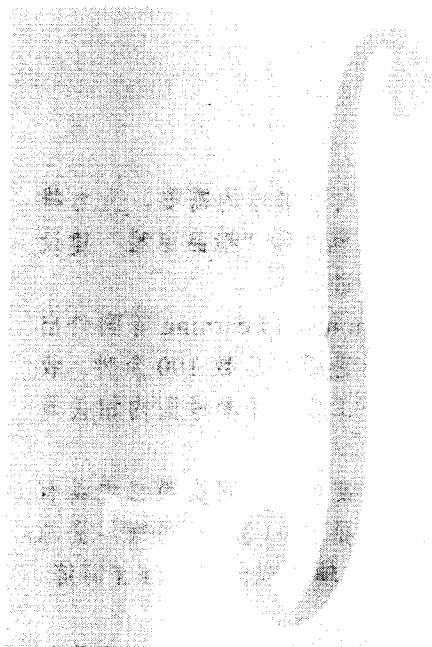
本系列丛书中,有 Finney、Weir 等编的《托马斯微积分》(第 10 版, Pearson),其特色可用“呈传统特色、富革新精神”概括,本书自 20 世纪 50 年代第 1 版以来,平均每四五年就有一个新版面世,长在 50 余年始终盛行于西方教坛,作者既有相当高的学术水平,又热爱教学,长期工作在教学第一线,其中,年近 90 的 G.B.Thomas 教授长年在 MIT 工作,具有丰富的教学经验;Finney 教授也在 MIT 工作达 10 年;Weir 是美国数学建模竞赛委员会主任。Stewart 编的立体化教材《微积分》(第 5 版, Thomson Learning)配备了丰富的教学资源,是国际上最畅销的微积分原版教材,2003 年全球销量约 40 余万册,在美国,占据了约 50%~60%的微积分教材市场,其用户包括耶鲁等名牌院校及众多一般院校 600 余所。本系列丛书还包括 Anton 编的经典教材《线性代数及其应用》(第 8 版, Wiley); Jay L. Devore 编的优秀教材《概率论与数理统计》(第 5 版, Thomson Learning)等。在努力降低引进教材售价方面,高等教育出版社做了大量和细致的工作,这套引进的教材体现了一定的权威性、系统性、先进性和经济性等特点。

通过影印、翻译、编译这批优秀教材,我们一方面要不断地分析、学习、消化吸收国外优秀教材的长处,吸取国外出版公司的制作经验,提升我们自编教材的立体化配套标准,使我国高校教材建设水平上一个新的台阶;与此同时,我们还将尝试组织海外作者和国内作者合编外文版基础课数学教材,并约请国内专家改编部分国外优秀教材,以适应我国实际教学环境。

这套教材出版后,我们将结合各高校的双语教学计划,开展大规模的宣传、培训工作,及时地将本套丛书推荐给高校使用。在使用过程中,我们衷心希望广大高校教师和同学提出宝贵的意见和建议,如有好的教材值得引进,请与高等教育出版社高等理科分社联系。

联系电话:010-58581384, E-mail: xuke@hep. com.cn.

高等教育出版社
2004 年 4 月 20 日



ABOUT THE COVER

The art on the cover was created by Bill Ralph, a mathematician who uses modern mathematics to produce visual representations of "dynamical systems."

Examples of dynamical systems in nature include the weather, blood pressure, the motions of the planets, and other phenomena that involve continual change. Such systems, which tend to be unpredictable and even chaotic at times, are modeled mathematically using the concepts of composition and iteration of functions.

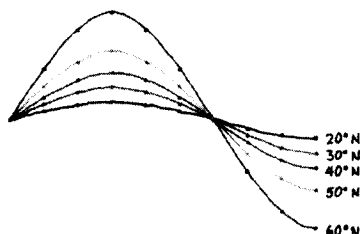
The process of creating the cover art starts with a photograph of a violin. The color values at each point on the photograph are then converted into numbers and a particular function is evaluated at each of those numbers giving a new number at each point of the photograph. The same function is then evaluated at each of these new numbers. Repeating this process produces a sequence of numbers called *iterates* of the function. The original photograph is then "repainted" using colors determined by certain properties of this sequence of iterates and the mathematical concept of "dimension." The final image is the result of mingling photographic reality with the complex behavior of a dynamical system.

CONTENTS

Preface	xiv
To the Student	xxvi

A Preview of Calculus 2

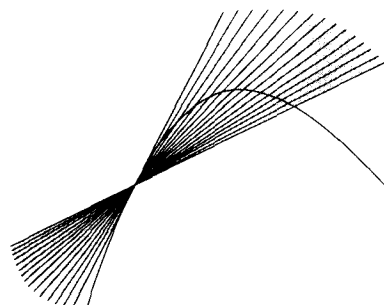
1 Functions and Models 10



1.1	Four Ways to Represent a Function	11
1.2	Mathematical Models: A Catalog of Essential Functions	25
1.3	New Functions from Old Functions	38
1.4	Graphing Calculators and Computers	48
1.5	Exponential Functions	55
1.6	Inverse Functions and Logarithms	63
	Review	77

Principles of Problem Solving	80
-------------------------------	----

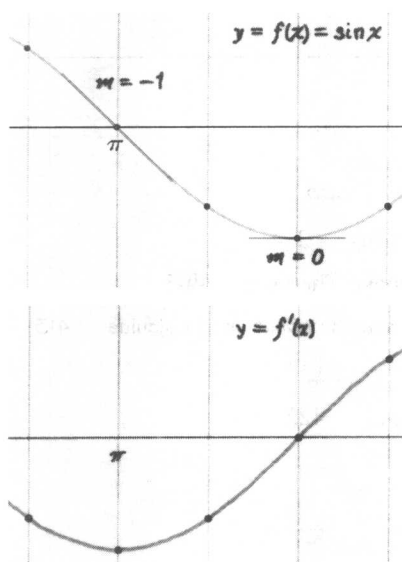
2 Limits and Derivatives 86



2.1	The Tangent and Velocity Problems	87
2.2	The Limit of a Function	92
2.3	Calculating Limits Using the Limit Laws	104
2.4	The Precise Definition of a Limit	114
2.5	Continuity	124
2.6	Limits at Infinity; Horizontal Asymptotes	135
2.7	Tangents, Velocities, and Other Rates of Change	149

2.8	Derivatives	158
	Writing Project • Early Methods for Finding Tangents	164
2.9	The Derivative as a Function	165
	Review	176
	Problems Plus	180

||| 3 Differentiation Rules 182



3.1	Derivatives of Polynomials and Exponential Functions	183
3.2	The Product and Quotient Rules	192
3.3	Rates of Change in the Natural and Social Sciences	199
3.4	Derivatives of Trigonometric Functions	211
3.5	The Chain Rule	217
3.6	Implicit Differentiation	227
3.7	Higher Derivatives	236
	Applied Project • Where Should a Pilot Start Descent?	243
	Applied Project • Building a Better Roller Coaster	243
3.8	Derivatives of Logarithmic Functions	244
3.9	Hyperbolic Functions	250
3.10	Related Rates	256
3.11	Linear Approximations and Differentials	262
	Laboratory Project • Taylor Polynomials	269
	Review	270
	Problems Plus	274

||| 4 Applications of Differentiation 278

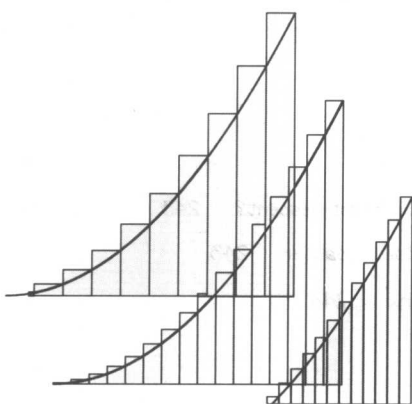


4.1	Maximum and Minimum Values	279
	Applied Project • The Calculus of Rainbows	288
4.2	The Mean Value Theorem	290
4.3	How Derivatives Affect the Shape of a Graph	296
4.4	Indeterminate Forms and L'Hospital's Rule	307
	Writing Project • The Origins of L'Hospital's Rule	315
4.5	Summary of Curve Sketching	316
4.6	Graphing with Calculus and Calculators	324

4.7	Optimization Problems	331
	Applied Project • The Shape of a Can	341
4.8	Applications to Business and Economics	342
4.9	Newton's Method	347
4.10	Antiderivatives	353
	Review	361
	Problems Plus	365

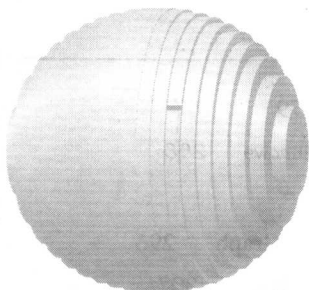
||| 5 Integrals 368

5.1	Areas and Distances	369
5.2	The Definite Integral	380
	Discovery Project • Area Functions	393
5.3	The Fundamental Theorem of Calculus	394
5.4	Indefinite Integrals and the Net Change Theorem	405
	Writing Project • Newton, Leibniz, and the Invention of Calculus	413
5.5	The Substitution Rule	414
5.6	The Logarithm Defined as an Integral	422
	Review	430
	Problems Plus	434

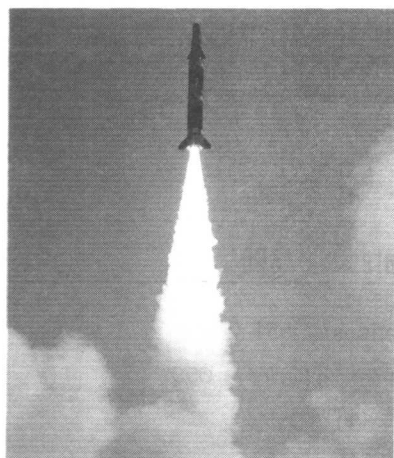


||| 6 Applications of Integration 436

6.1	Areas between Curves	437
6.2	Volumes	444
6.3	Volumes by Cylindrical Shells	455
6.4	Work	460
6.5	Average Value of a Function	464
	Applied Project • Where to Sit at the Movies	468
	Review	468
	Problems Plus	470

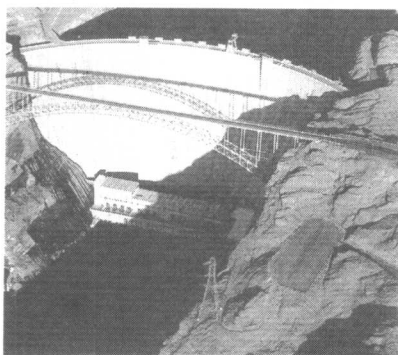


||| 7 Techniques of Integration 474



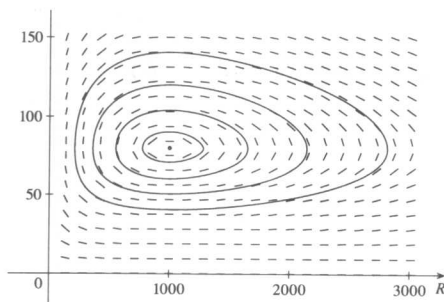
- 7.1 Integration by Parts 475
- 7.2 Trigonometric Integrals 482
- 7.3 Trigonometric Substitution 489
- 7.4 Integration of Rational Functions by Partial Fractions 496
- 7.5 Strategy for Integration 505
- 7.6 Integration Using Tables and Computer Algebra Systems 511
 - Discovery Project • Patterns in Integrals 517
- 7.7 Approximate Integration 518
- 7.8 Improper Integrals 530
 - Review 540
- Problems Plus 543

||| 8 Further Applications of Integration 546



- 8.1 Arc Length 547
 - Discovery Project • Arc Length Contest 554
- 8.2 Area of a Surface of Revolution 554
 - Discovery Project • Rotating on a Slant 560
- 8.3 Applications to Physics and Engineering 561
- 8.4 Applications to Economics and Biology 571
- 8.5 Probability 575
 - Review 582
- Problems Plus 584

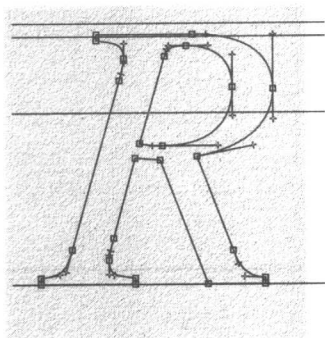
||| 9 Differential Equations 586



- 9.1 Modeling with Differential Equations 587
- 9.2 Direction Fields and Euler's Method 592
- 9.3 Separable Equations 601
 - Applied Project • How Fast Does a Tank Drain? 609
 - Applied Project • Which Is Faster, Going Up or Coming Down? 610
- 9.4 Exponential Growth and Decay 611
 - Applied Project • Calculus and Baseball 622

9.5	The Logistic Equation	623
9.6	Linear Equations	632
9.7	Predator-Prey Systems	638
	Review	644

Problems Plus 648



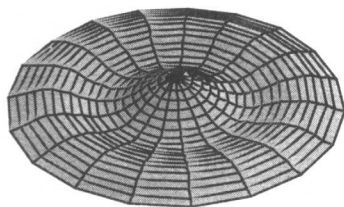
||| 10 Parametric Equations and Polar Coordinates 650

10.1	Curves Defined by Parametric Equations	651
	Laboratory Project • Running Circles around Circles	659
10.2	Calculus with Parametric Curves	660
	Laboratory Project • Bézier Curves	669
10.3	Polar Coordinates	669
10.4	Areas and Lengths in Polar Coordinates	679
10.5	Conic Sections	684
10.6	Conic Sections in Polar Coordinates	692
	Review	696

Problems Plus 699

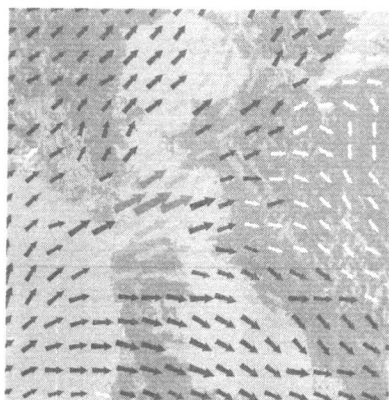
||| 11 Infinite Sequences and Series 700

11.1	Sequences	701
	Laboratory Project • Logistic Sequences	713
11.2	Series	713
11.3	The Integral Test and Estimates of Sums	723
11.4	The Comparison Tests	730
11.5	Alternating Series	735
11.6	Absolute Convergence and the Ratio and Root Tests	740
11.7	Strategy for Testing Series	747
11.8	Power Series	749
11.9	Representations of Functions as Power Series	754



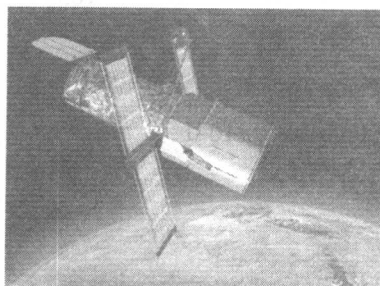
11.10	Taylor and Maclaurin Series	760
	Laboratory Project • An Elusive Limit	772
11.11	The Binomial Series	772
	Writing Project • How Newton Discovered the Binomial Series	776
11.12	Applications of Taylor Polynomials	776
	Applied Project • Radiation from the Stars	785
	Review	786
	Problems Plus	789

||| 12 Vectors and the Geometry of Space 792



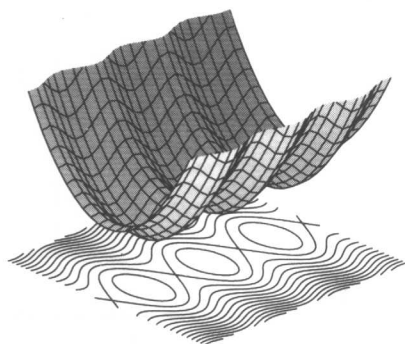
12.1	Three-Dimensional Coordinate Systems	793
12.2	Vectors	798
12.3	The Dot Product	807
12.4	The Cross Product	814
	Discovery Project • The Geometry of a Tetrahedron	822
12.5	Equations of Lines and Planes	822
	Laboratory Project • Putting 3D in Perspective	832
12.6	Cylinders and Quadric Surfaces	832
12.7	Cylindrical and Spherical Coordinates	839
	Laboratory Project • Families of Surfaces	844
	Review	844
	Problems Plus	847

||| 13 Vector Functions 848



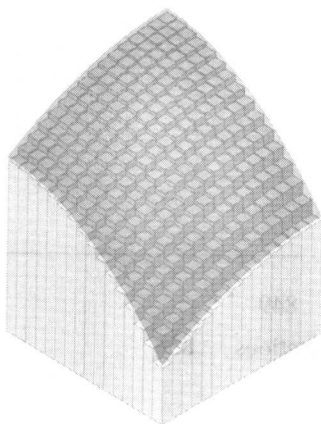
13.1	Vector Functions and Space Curves	849
13.2	Derivatives and Integrals of Vector Functions	856
13.3	Arc Length and Curvature	862
13.4	Motion in Space: Velocity and Acceleration	870
	Applied Project • Kepler's Laws	880
	Review	881
	Problems Plus	884

||| 14 Partial Derivatives 886



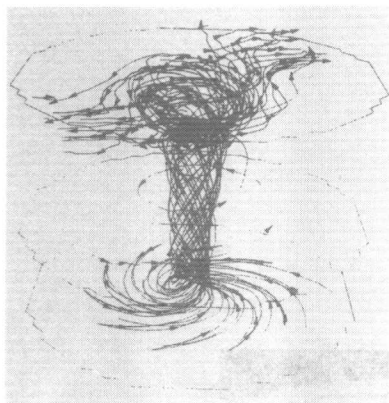
- 14.1 Functions of Several Variables 887
- 14.2 Limits and Continuity 902
- 14.3 Partial Derivatives 909
- 14.4 Tangent Planes and Linear Approximations 923
- 14.5 The Chain Rule 931
- 14.6 Directional Derivatives and the Gradient Vector 940
- 14.7 Maximum and Minimum Values 953
 - Applied Project • Designing a Dumpster 963
 - Discovery Project • Quadratic Approximations and Critical Points 964
- 14.8 Lagrange Multipliers 965
 - Applied Project • Rocket Science 972
 - Applied Project • Hydro-Turbine Optimization 973
- Review 974
- Problems Plus 978

||| 15 Multiple Integrals 980



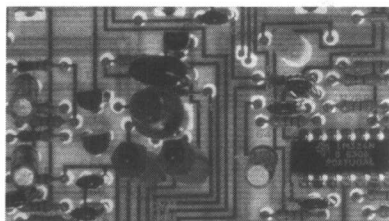
- 15.1 Double Integrals over Rectangles 981
- 15.2 Iterated Integrals 989
- 15.3 Double Integrals over General Regions 995
- 15.4 Double Integrals in Polar Coordinates 1003
- 15.5 Applications of Double Integrals 1009
- 15.6 Surface Area 1019
- 15.7 Triple Integrals 1023
 - Discovery Project • Volumes of Hyperspheres 1032
- 15.8 Triple Integrals in Cylindrical and Spherical Coordinates 1033
 - Applied Project • Roller Derby 1039
 - Discovery Project • The Intersection of Three Cylinders 1040
- 15.9 Change of Variables in Multiple Integrals 1041
 - Review 1049
- Problems Plus 1052

||| 16 Vector Calculus 1054



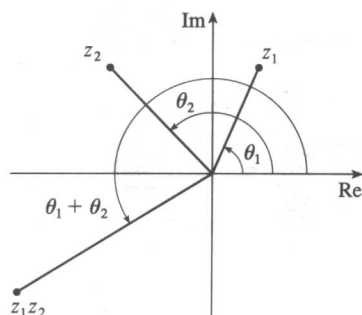
16.1	Vector Fields	1055
16.2	Line Integrals	1062
16.3	The Fundamental Theorem for Line Integrals	1074
16.4	Green's Theorem	1083
16.5	Curl and Divergence	1090
16.6	Parametric Surfaces and Their Areas	1098
16.7	Surface Integrals	1109
16.8	Stokes' Theorem	1121
	Writing Project • Three Men and Two Theorems	1126
16.9	The Divergence Theorem	1127
16.10	Summary	1134
	Review	1135
	Problems Plus	1138

||| 17 Second-Order Differential Equations 1140



17.1	Second-Order Linear Equations	1141
17.2	Nonhomogeneous Linear Equations	1147
17.3	Applications of Second-Order Differential Equations	1155
17.4	Series Solutions	1163
	Review	1167

||| Appendixes A1



A	Numbers, Inequalities, and Absolute Values	A2
B	Coordinate Geometry and Lines	A10
C	Graphs of Second-Degree Equations	A16
D	Trigonometry	A24
E	Sigma Notation	A34
F	Proofs of Theorems	A39
G	Complex Numbers	A49
H	Answers to Odd-Numbered Exercises	A57

||| Index A125

Infinite sequences and series were introduced briefly in A Preview of Calculus in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2} . (Recall that we have previously been unable to do this.) Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.12. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \dots, a_n, \dots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1} .

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n .

NOTATION • The sequence $\{a_1, a_2, a_3, \dots\}$ is also denoted by

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1 Some sequences can be defined by giving a formula for the n th term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

$$\begin{array}{lll} \text{(a)} & \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} & a_n = \frac{n}{n+1} & \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots \right\} \\ \text{(b)} & \left\{ \frac{(-1)^n(n+1)}{3^n} \right\} & a_n = \frac{(-1)^n(n+1)}{3^n} & \left\{ -\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots \right\} \end{array}$$

$$(c) \quad \left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \quad a_n = \sqrt{n-3}, \quad n \geq 3 \quad \{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$$

$$(d) \quad \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \quad a_n = \cos \frac{n\pi}{6}, \quad n \geq 0 \quad \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos \frac{n\pi}{6}, \dots \right\}$$

EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5} \quad a_2 = -\frac{4}{25} \quad a_3 = \frac{5}{125} \quad a_4 = -\frac{6}{625} \quad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the n th term will have numerator $n + 2$. The denominators are the powers of 5, so a_n has denominator 5^n . The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1 . In Example 1(b) the factor $(-1)^n$ meant we started with a negative term. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore,

$$a_n = (-1)^{n-1} \frac{n+2}{5^n}$$

EXAMPLE 3 Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n .

(b) If we let a_n be the digit in the n th decimal place of the number e , then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \dots\}$$

(c) The **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2} \quad n \geq 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 65).

A sequence such as the one in Example 1(a), $a_n = n/(n+1)$, can be pictured either by plotting its terms on a number line as in Figure 1 or by plotting its graph as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \quad (2, a_2) \quad (3, a_3) \quad \dots \quad (n, a_n) \quad \dots$$

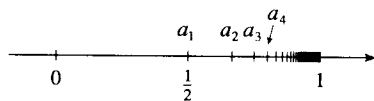


FIGURE 1

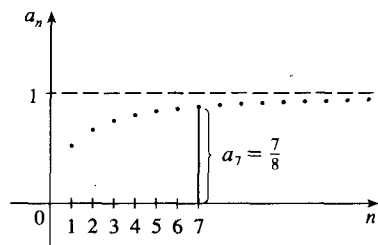


FIGURE 2

From Figure 1 or 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n \rightarrow \infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 2.6.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \rightarrow \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L .

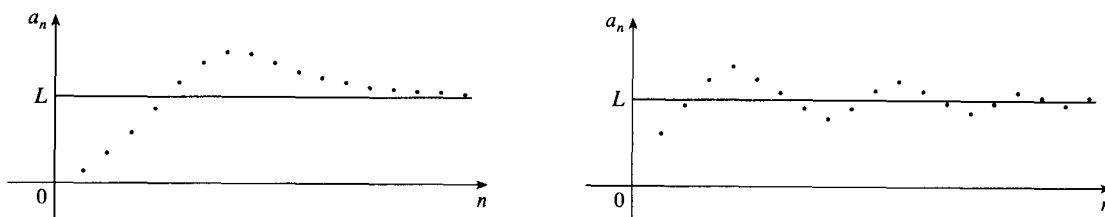


FIGURE 3
Graphs of two
sequences with
 $\lim_{n \rightarrow \infty} a_n = L$

A more precise version of Definition 1 is as follows.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \rightarrow \infty} a_n = L \quad \text{or} \quad a_n \rightarrow L \text{ as } n \rightarrow \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \quad \text{whenever } n > N$$

||| Compare this definition with Definition 2.6.7.

Definition 2 is illustrated by Figure 4, in which the terms a_1, a_2, a_3, \dots are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

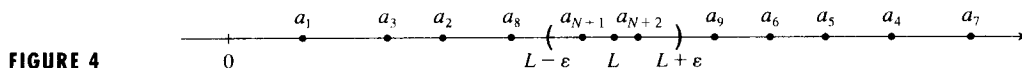


FIGURE 4

Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N .

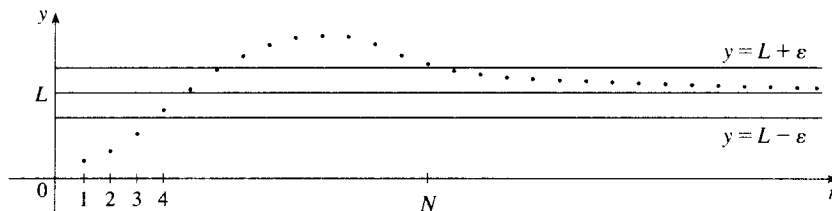


FIGURE 5

Comparison of Definition 2 and Definition 2.6.7 shows that the only difference between $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{x \rightarrow \infty} f(x) = L$ is that n is required to be an integer. Thus, we have the following theorem, which is illustrated by Figure 6.

3 Theorem If $\lim_{x \rightarrow \infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n \rightarrow \infty} a_n = L$.

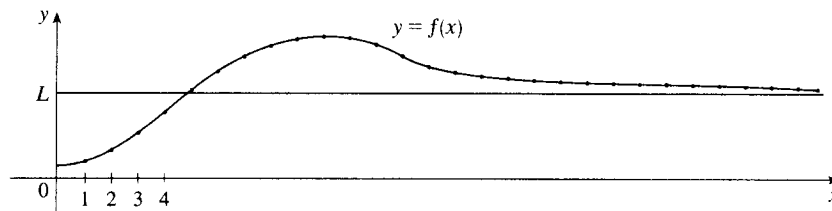


FIGURE 6

In particular, since we know that $\lim_{x \rightarrow \infty} (1/x^r) = 0$ when $r > 0$ (Theorem 2.6.5), we have

$$\boxed{4} \quad \lim_{n \rightarrow \infty} \frac{1}{n^r} = 0 \quad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n \rightarrow \infty} a_n = \infty$. The following precise definition is similar to Definition 2.6.9.

5 Definition $\lim_{n \rightarrow \infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \quad \text{whenever } n > N$$

If $\lim_{n \rightarrow \infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞ .