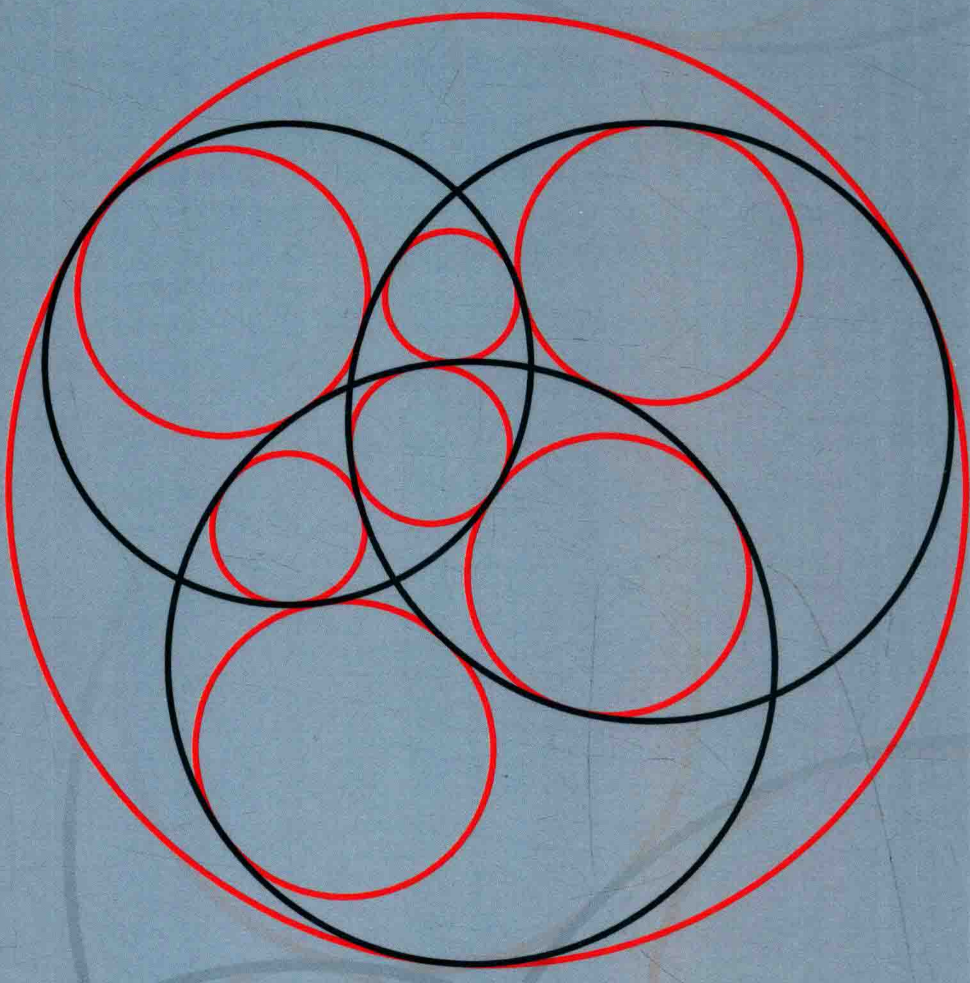
3264 AND ALL THAT

A SECOND COURSE IN ALGEBRAIC GEOMETRY

DAVID EISENBUD AND JOE HARRIS



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The enumeration of solutions to systems of polynomial equations in several variables has been an active area of mathematics since the early work of Leibniz. In the 19th century, Chasles calculated that there are 3264 smooth conic plane curves tangent to five given general conics – a landmark in the field and perhaps the first important "excess intersection" problem.

Such computations in intersection theory were part of the motivation of Poincaré's development of topology, and also figured in Hilbert's Problems from 1900. Since then, intersection theory has become a topic of central importance in mathematics, with applications to topology, number theory and mathematical physics.

This book can form the basis of a second course in algebraic geometry. As motivation, it takes concrete problems from enumerative geometry and intersection theory. Its aim is to provide intuition and technique so that the student develops the ability to solve geometric problems.

The authors explain and illustrate key ideas such as rational equivalence, Chow rings, Grassmanians, Schubert calculus and Chern classes, excess intersection theory and the Grothendieck Riemann–Roch theorem. The geometric applications range from the 27 lines on a cubic surface through the existence of special divisors on Riemann surfaces.

Readers will appreciate the abundance of examples, many provided as exercises with solutions available online.

Preface

We have been working on this project for over ten years, and at times we have felt that we have only brought on ourselves a plague of locus. However, our spirits have been lightened, and the project made far easier and more successful than it would have been, by the interest and help of many people.

First of all, we thank Bill Fulton, who created much of the modern approach to intersection theory, and who directly informed our view of the subject from the beginning.

Many people have helped us by reading early versions of the text and providing comments and corrections. Foremost among these is Paolo Aluffi, who gave extensive and detailed comments; we also benefited greatly from the advice of Francesco Cavazzani and Izzet Coşkun. We would also thank Mike Roth and Stephanie Yang, who provided notes on the early iterations of a course on which much of this text is based, as well as students who contributed corrections, including Sitan Chen, Jun Hou Fung, Changho Han, Chi-Yun Hsu, Hannah Larson, Ravi Jagadeesan, Aaron Landesman, Yogesh More, Arpon Raksit, Ashvin Swaminathan, Arnav Tripathy, Isabel Vogt and Lynnelle Ye.

Silvio Levy made many of the many illustrations in this book (and occasionally corrected our mathematical errors too!). Devlin Mallory then took over as copyeditor, and completed the rest of the figures. We are grateful to both of them for their many improvements to this text (and to Cambridge University Press for hiring Devlin!).

We are all familiar with the after-the-fact tone — weary, self-justificatory, aggrieved, apologetic — shared by ship captains appearing before boards of inquiry to explain how they came to run their vessels aground, and by authors composing forewords.

–John Lanchester

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Chapter 0

Introduction

Es gibt nach des Verf. Erfarhrung kein besseres Mittel, Geometrie zu lernen, als das Studium des Schubertschen *Kalküls der abzählenden Geometrie.*

(There is, in the author's experience, no better means of learning geometry than the study of Schubert's *Calculus of Enumerative Geometry.*)

–B. L. van der Waerden (in a Zentralblatt review of *An Introduction to Enumerative Geometry* by Hendrik de Vries).

Why you want to read this book

Algebraic geometry is one of the central subjects of mathematics. All but the most analytic of number theorists speak our language, as do mathematical physicists, complex analysts, homotopy theorists, symplectic geometers, representation theorists.... How else could you get between such apparently disparate fields as topology and number theory in one hop, except via algebraic geometry?

And intersection theory is at the heart of algebraic geometry. From the very beginnings of the subject, the fact that the number of solutions to a system of polynomial equations is, in many circumstances, constant as we vary the coefficients of those polynomials has fascinated algebraic geometers. The distant extensions of this idea still drive the field forward.

At the outset of the 19th century, it was to extend this "preservation of number" that algebraic geometers made two important choices: to work over the complex numbers rather than the real numbers, and to work in projective space rather than affine space. (With these choices the two points of intersection of a line and an ellipse have somewhere to go as the ellipse moves away from the real points of the line, and the same for the point of intersection of two lines as the lines become parallel.) Over the course of the century, geometers refined the art of counting solutions to geometric problems — introducing the central notion of a *parameter space*, proposing the notions of an equivalence relation