WENTWORTH-SMITH MATHEMATICAL SERIES

PLANE GEOMETRY

BY

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AND

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PREFACE

Long after the death of Robert Recorde, England's first great writer of textbooks, the preface of a new edition of one of his works contained the appreciative statement that the book was "entail'd upon the People, ratified and sign'd by the approbation of Time." The language of this sentiment sounds quaint, but the noble tribute is as impressive to-day as when first put in print two hundred fifty years ago.

With equal truth these words may be applied to the Geometry written by George A. Wentworth. For a generation it has been the leading textbook on the subject in America. It set a standard for usability that every subsequent writer upon geometry has tried to follow, and the number of pupils who have testified to its excellence has run well into the millions.

In undertaking to prepare a work to take the place of the Wentworth Geometry, the authors have been guided by certain well-defined principles, based upon an extended investigation of the needs of the schools and upon a study of all that is best in the recent literature of the subject. The effects of these principles they feel should be summarized for the purpose of calling the attention of the wide circle of friends of the Wentworth-Smith series to the points of similarity and of difference in the two works.

1. Every effort has been made not only to preserve but to improve upon the simplicity of treatment, the clearness of expression, and the symmetry of page that characterized the successive editions of the Wentworth Geometry. It has been the purpose to prepare a book that should do even more than maintain the traditions this work has fostered.

2. The proofs have been given substantially in full, to the end that the pupil may always have before him a model for his independent treatment of the exercises.

3. The sequence of propositions has been improved in several respects, notably in the treatment of parallels.

4. To meet a general demand, the number of propositions has been decreased so as to include only the great basal theorems and problems. A little of the less important material has been placed in the Appendix, to be used or not as circumstances demand.

5. The exercises, in some respects the most important part of a course in geometry, have been rendered more dignified in appearance and have been improved in content. The number of simple exercises has been greatly increased, while the difficult puzzle is much less in evidence than in most American textbooks. The exercises are systematically grouped, appearing in full pages, in large type, at frequent intervals. They are not all intended for one class, but are so numerous as to allow the teacher to make selections from year to year.

6. The introduction has been made as concrete as is reasonable. Definitions have been postponed until they are actually needed, only well-recognized terms have been employed, the pupil is initiated at once into the practical use of the instruments, some of the reasons for studying geometry are early shown in an interesting way, and correlation is made with the simple algebra already studied.

The authors are indebted to many friends of the Wentworth-Smith series for assistance and encouragement in the labor of preparing this work, and they will welcome any further suggestions for improvement from any of their readers.

GEORGE WENTWORTH
DAVID EUGENE SMITH

SYMBOLS AND ABBREVIATIONS

$=$	equals, equal, equal to, is equal to, or is equivalent to.	Adj.	adjacent.
$>$	is greater than.	Alt.	alternate.
$<$	is less than.	Ax.	axiom.
$\parallel$	parallel.	Const.	construction.
$\perp$	perpendicular.	Cor.	corollary.
$\angle$	angle.	Def.	definition.
$\triangle$	triangle.	Ex.	exercise.
▱	parallelogram.	Ext.	exterior.
□	rectangle.	Fig.	figure.
$\odot$	circle.	Hyp.	hypothesis.
st.	straight.	Iden.	identity.
rt.	right.	Int.	interior.
$\because$	since.	Post.	postulate.
$\therefore$	therefore.	Prob.	problem.
		Prop.	proposition.
		Sup.	supplementary.

These symbols take the plural form when necessary, as in the case of ‖s, ∠s, △s, ⊙s.

The symbols $+$, $-$, $\times$, $\div$ are used as in algebra.

There is no generally accepted symbol for "is congruent to," and the words are used in this book. Some teachers use $\cong$ or $\simeq$, and some use $\equiv$, but the sign of equality is more commonly employed, the context telling whether equality, equivalence, or congruence is to be understood.

Q. E. D. is an abbreviation that has long been used in geometry for the Latin words *quod erat demonstrandum*, "which was to be proved."

Q. E. F. stands for *quod erat faciendum*, "which was to be done."

CONTENTS

PLANE GEOMETRY

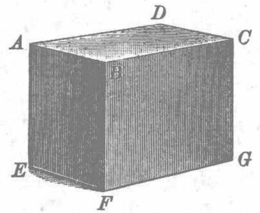
INTRODUCTION

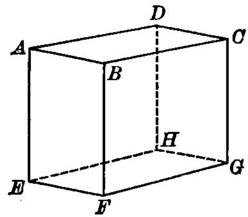
1. The Nature of Arithmetic. In arithmetic we study computation, the working with numbers. We may have a formula expressed in algebraic symbols, such as $a = bh$, where a may stand for the area of a rectangle, and b and h respectively for the number of units of length in the base and height; but the actual computation involved in applying such formula to a particular case is part of arithmetic.

2. The Nature of Algebra. In algebra we generalize the arithmetic, and instead of saying that the area of a rectangle with base 4 in. and height 2 in. is 4×2 sq. in., we express a general law by saying that $a = bh$. In arithmetic we may have an equality, like $2 \times 16 + 17 = 49$, but in algebra we make much use of equations, like $2x + 17 = 49$. Algebra, therefore, is a generalized arithmetic.

3. The Nature of Geometry. We are now about to begin another branch of mathematics, one not chiefly relating to numbers although it uses numbers, and not primarily devoted to equations although using them, but one that is concerned principally with the study of forms, such as triangles, parallelograms, and circles. Many facts that are stated in arithmetic and algebra are proved in geometry. For example, in geometry it is proved that the square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides, and that the circumference of a circle equals 3.1416 times the diameter.

4. Solid. The block here represented is called a *solid;* it is a limited portion of space filled with matter. In geometry, however, we have nothing to do with the matter of which a





body is composed; we study simply its *shape* and *size*, as in the second figure.

That is, a physical solid can be touched and handled; a geometric solid is the space that a physical solid is conceived to occupy. For example, a stick is a physical solid; but if we put it into wet plaster, and then remove it, the hole that is left may be thought of as a geometric solid although it is filled with air.

5. Geometric Solid. A limited portion of space is called a *geometric solid.*

6. Dimensions. The block represented in § 4 extends in three principal directions:

(1) From left to right, that is, from A to D;
(2) From back to front, that is, from A to B;
(3) From top to bottom, that is, from A to E.

These extensions are called the *dimensions* of the block, and are named in the order given, *length, breadth* (or width), and *thickness* (height, altitude, or depth). Similarly, we may say that every solid has three dimensions.

Very often a solid is of such shape that we cannot point out the length, or distinguish it from the breadth or thickness, as an irregular block of coal. In the case of a round ball, where the length, breadth, and thickness are all the same in extent, it is impossible to distinguish one dimension from the others.

7. Surface. The block shown in § 4 has six flat faces, each of which is called a *surface.* If the faces are made smooth by polishing, so that when a straight edge is applied to any one of them the straight edge in every part will touch the surface, each face is called a *plane surface,* or a *plane.*

These surfaces are simply the boundaries of the solid. They have no thickness, even as a colored light shining upon a piece of paper does not make the paper thicker. A board may be planed thinner and thinner, and then sandpapered still thinner, thus coming nearer and nearer to representing what we think of as a geometric plane, but it is always a solid bounded by surfaces.

That which has length and breadth without thickness is called a *surface.*

8. Line. In the solid shown in § 4 we see that two adjacent surfaces intersect in a line. A line is therefore simply the boundary of a surface, and has neither breadth nor thickness.

That which has length without breadth or thickness is called a *line.*

A telegraph wire, for example, is not a line. It is a solid. Even a pencil mark has width and a very little thickness, so that it is also a solid. But if we think of a wire as drawn out so that it becomes finer and finer, it comes nearer and nearer to representing what we think of and speak of as a geometric line.

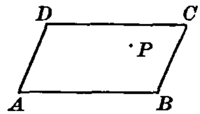
9. Magnitudes. Solids, surfaces, and lines are called *magnitudes.*

10. Point. In the solid shown in § 4 we see that when two lines meet they meet in a point. A point is therefore simply the boundary of a line, and has no length, no breadth, and no thickness.

That which has only position, without length, breadth, or thickness, is called a *point.*

We may think of the extremity of a line as a point. We may also think of the intersection of two lines as a point, and of the intersection of two surfaces as a line.

11. Representing Points and Geometric Magnitudes. Although we only imagine such geometric magnitudes as lines or planes, we may represent them by pictures.



Thus we represent a point by a fine dot, and name it by a letter, as P in this figure.

We represent a line by a fine mark, and name it by letters placed at the ends, as AB.

We represent a surface by its boundary lines, and name it by letters placed at the corners or in some other convenient way, as $ABCD$.

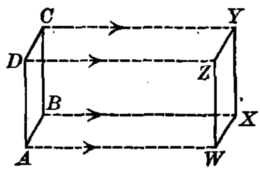
We represent a solid by the boundary faces or by the lines bounding the faces, as in § 4.

12. Generation of Geometric Magnitudes. We may think of

(1) A line as generated by a moving point;
(2) A surface as generated by a moving line;
(3) A solid as generated by a moving surface.

For example, as shown in the figure let the surface $ABCD$ move to the position $WXYZ$. Then

(1) A generates the line AW;
(2) AB generates the surface $AWXB$;
(3) $ABCD$ generates the solid AY.



Of course a point will not generate a line by simply turning over, for this is not motion for a point; nor will a line generate a surface by simply sliding along itself; nor will a surface generate a solid by simply sliding upon itself.

13. Geometric Figure. A point, a line, a surface, a solid, or any combination of these, is called a *geometric figure.*

A geometric figure is generally called simply a *figure.*

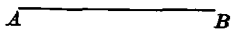
14. Geometry. The science of geometric figures is called *geometry.*

Plane geometry treats of figures that lie wholly in the same plane, that is, of plane figures.

Solid geometry treats of figures that do not lie wholly in the same plane.

15. Straight Line. A line such that any part placed with its ends on any other part must lie wholly in the line is called a *straight line.*

For example, AB is a straight line, for if we take, say, a half inch of it, and place it in any way on any other part of AB, but so that its ends lie in AB, then the whole of the half inch of line will lie in AB. This is well shown by using tracing paper. The word *line* used alone is understood to mean a straight line.



Part of a straight line is called a *segment* of the line. The term *segment* is applied also to certain other magnitudes.

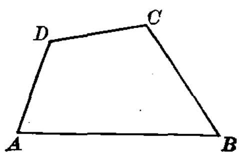
16. Equality of Lines. Two straight-line segments that can be placed one upon the other so that their extremities coincide are said to be *equal.*

In general, two geometric magnitudes are equal if they can be made to coincide throughout their whole extent. We shall see later that some figures that coincide are said to be *congruent.*

17. Broken Line. A line made up of two or more different straight lines is called a *broken line.*

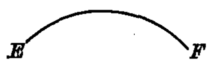
For example, CD is a broken line.

18. Rectilinear Figure. A plane figure formed by a broken line is called a *rectilinear figure.*



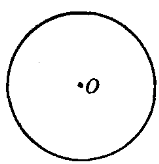
For example, $ABCD$ is a rectilinear figure.

19. Curve Line. A line no part of which is straight is called a *curve line,* or simply a *curve.*



For example, EF is a curve line.

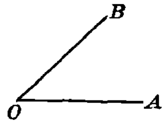
20. Curvilinear Figure. A plane figure formed by a curve line is called a *curvilinear figure.*



For example, O is a curvilinear figure with which we are already familiar.

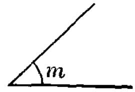
Some curvilinear figures are surfaces bounded by curves and others are the curves themselves.

21. Angle. The opening between two straight lines drawn from the same point is called an *angle*.



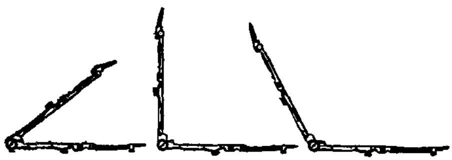
Strictly speaking, this is a *plane* angle. We shall find later that there are angles made by curve lines and angles made by planes.

The two lines are called the *sides* of the angle, and the point of meeting is called the *vertex*.



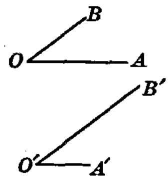
An angle may be read by naming the letters designating the sides, the vertex letter being between the others, as the angle AOB. An angle may also be designated by the vertex letter, as the angle O, or by a small letter within, as the angle m. A curve is often drawn to show the particular angle meant, as in angle m.

22. Size of Angle. The size of an angle depends upon the amount of turning necessary to bring one side into the position of the other.



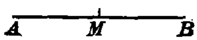
One angle is greater than another angle when the amount of turning is greater. Thus in these compasses the first angle is smaller than the second, which is also smaller than the third. The length of the sides has nothing to do with the size of the angle.

23. Equality of Angles. Two angles that can be placed one upon the other so that their vertices coincide and the sides of one lie along the sides of the other are said to be *equal*.



For example, the angles AOB and $A'O'B'$ (read "A prime, O prime, B prime") are equal. It is well to illustrate this by tracing one on thin paper and placing it upon the other.

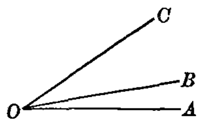
24. Bisector. A point, a line, or a plane that divides a geometric magnitude into two equal parts is called a *bisector* of the magnitude.



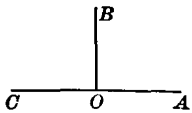
For example, M, the mid-point of the line AB, is a bisector of the line.

25. Adjacent Angles. Two angles that have the same vertex and a common side between them are called *adjacent angles*.

For example, the angles AOB and BOC are adjacent angles, and in § 26 the angles AOB and BOC are adjacent angles.



26. Right Angle. When one straight line meets another straight line and makes the adjacent angles equal, each angle is called a *right angle.*

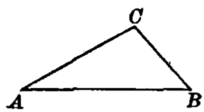


For example, angles AOB and BOC in this figure. If CO is cut off, angle AOB is still a right angle.

27. Perpendicular. A straight line making a right angle with another straight line is said to be *perpendicular* to it.

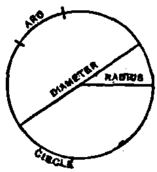
Thus OB is perpendicular to CA, and CA to OB. OB is also called *a perpendicular* to CA, and O is called the *foot* of the perpendicular OB.

28. Triangle. A portion of a plane bounded by three straight lines is called a *triangle.*



The lines AB, BC, and CA are called the ***sides*** of the triangle ABC, and the sides taken together form the *perimeter*. The points A, B, and C are the *vertices* of the triangle, and the angles A, B, and C are the ***angles*** of the triangle. The side AB upon which the triangle is supposed to rest is the *base* of the triangle. Similarly for other plane figures.

29. Circle. A closed curve lying in a plane, and such that all of its points are equally distant from a fixed point in the plane, is called a *circle.*

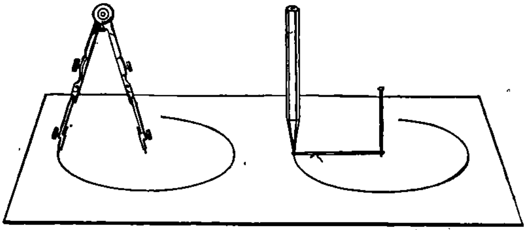


The length of the circle is called the *circumference.* The point from which all points on the circle are equally distant is the *center*. Any portion of a circle is an *arc*. A straight line from the center to the circle is a *radius*. A straight line through the center, terminated at each end by the circle, is a *diameter*.

Formerly in elementary geometry *circle* was taken to mean the space inclosed, and the bounding line was called the circumference. Modern usage has conformed to the definition used in higher mathematics.

30. Instruments of Geometry. In geometry only two instruments are necessary besides pencil and paper. These are a straight edge, or ruler, and a pair of compasses.

It is evident that *all radii of the same circle are equal.*



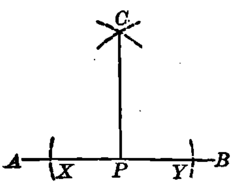
In the absence of compasses, and particularly for blackboard work, a loop made of string may be used. For the accurate transfer of lengths, however, compasses are desirable.

31. Exercises in using Instruments. The following simple exercises are designed to accustom the pupil to the use of instruments. No proofs are attempted, these coming later in the course.

This section may be omitted if desired, without affecting the course.

EXERCISE 1

1. From a given point on a given straight line required to draw a perpendicular to the line.



Let AB be the given line and P be the given point.

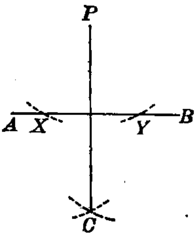
It is required to draw from P a line perpendicular to AB.

With P as a center and any convenient radius draw arcs cutting AB at X and Y.

With X as a center and XY as a radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straight edge draw a line from P to C, and this will be the perpendicular required.

2. From a given point outside a given straight line required to let fall a perpendicular to the line.



Let AB be the given straight line and P be the given point.

It is required to draw from P a line perpendicular to AB.

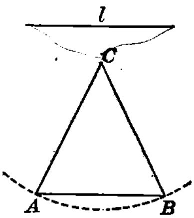
With P as a center and any convenient radius draw an arc cutting AB at X and Y.

With X as a center and any convenient radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straight edge draw a straight line from P to C, and this will be the perpendicular required.

It is interesting to test the results in Exs. 1 and 2, by cutting the paper and fitting the angles together.

3. Required to draw a triangle having two sides each equal to a given line.



Let l be the given line.

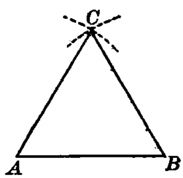
It is required to draw a triangle having two sides each equal to l.

With any center, as C, and a radius equal to l draw an arc.

Join any two points on the arc, as A and B, with each other and with C by straight lines.

Then ABC is the triangle required.

4. Required to draw a triangle having its three sides each equal to a given line.



Let AB be the given line.

It is required to draw a triangle having its three sides each equal to AB.

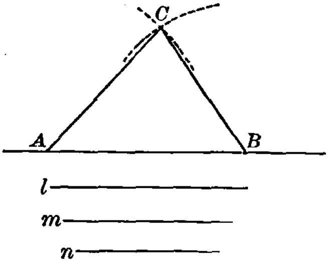
With A as a center and AB as a radius draw a circle, and with B as a center and the same radius draw another circle.

Join either intersection of the circles with A and B by straight lines.

Then ABC is the triangle required.

In such cases draw the arcs only long enough to show the point of intersection.

5. Required to draw a triangle having its sides equal respectively to three given lines.



Let the three lines be l, m, and n.

What is now required?

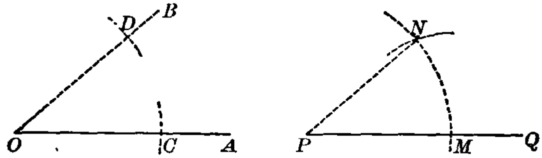
Upon any line mark off with the compasses a line-segment AB equal to l.

With A as a center and m as a radius draw a circle; with B as a center and n as a radius draw a circle.

Draw AC and BC.

Then ABC is the required triangle.

6. From a given point on a given line required to draw a line making an angle equal to a given angle.



Let P be the given point on the given line PQ, and let angle AOB be the given angle.

What is now required?

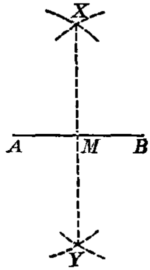
With O as a center and any radius draw an arc cutting AO at C and BO at D.

With P as a center and OC as a radius draw an arc cutting PQ at M.

With M as a center and the straight line joining C and D as a radius draw an arc cutting the arc just drawn at N, and draw PN.

Then angle MPN is the required angle.

7. Required to bisect a given straight line.



Let AB be the given line.

It is required to bisect AB.

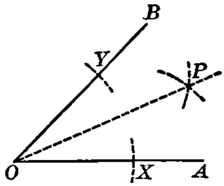
With A as a center and AB as a radius draw a circle, and with B as a center and the same radius draw a circle.

Call the two intersections of the circles X and Y.

Draw the straight line XY.

Then XY bisects the line AB at the point of intersection M.

8. Required to bisect a given angle.



Let AOB be the given angle.

It is required to bisect the angle AOB.

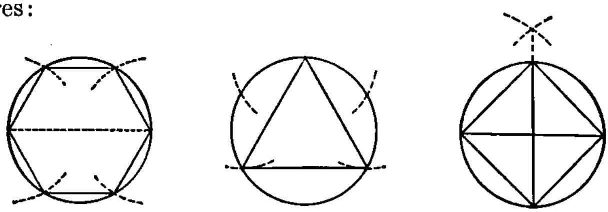
With O as a center and any convenient radius draw an arc cutting OA at X and OB at Y.

With X as a center and a line joining X and Y as a radius draw a circle, and with Y as a center and the same radius draw a circle, and call one point of intersection of the circles P.

Draw the straight line OP.

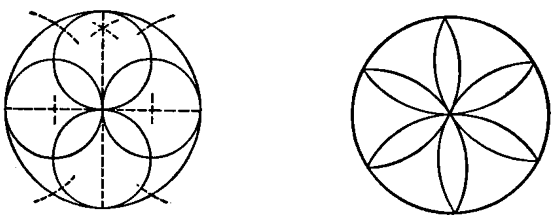
Then OP is the required bisector.

9. By the use of compasses and ruler draw the following figures:



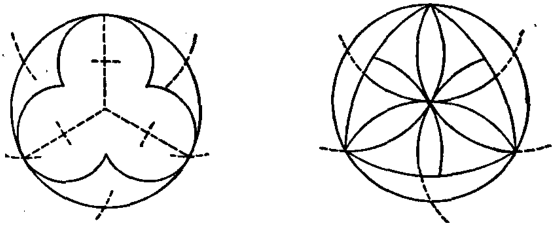
The dotted lines show how to fix the points needed in drawing the figure, and they may be erased after the figure is completed. In general, in geometry, auxiliary lines (those needed only as aids) are indicated by dotted lines.

10. By the use of compasses and ruler draw the following figures:

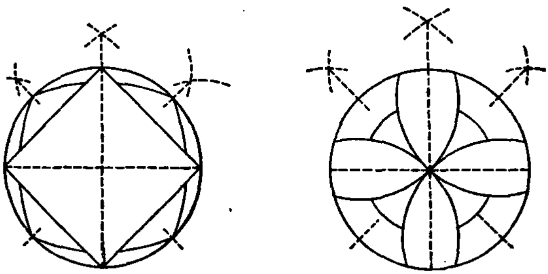


It is apparent from the figures in Exs. 9 and 10 that the radius of the circle may be used in describing arcs that shall divide the circle into six equal parts.

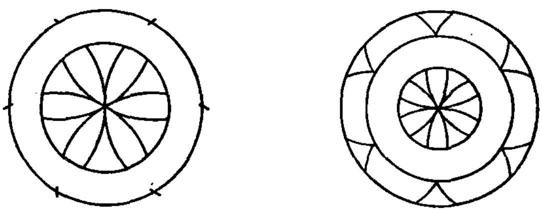
11. By the use of compasses and ruler draw the following figures:



12. By the use of compasses and ruler draw the following figures:



13. By the use of compasses and ruler draw the following figures:



In such figures artistic patterns may be made by coloring various portions of the drawings. In this way designs are made for stained-glass windows, for oilcloth, for colored tiles, and for other decorations.

14. Draw a triangle of which each side is $1\frac{1}{4}$ in.

15. Draw two lines bisecting each other at right angles.