WAVE PROPAGATION
AND
GROUP VELOCITY

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FOREWORD

Munich, in the spring 1913, was a very lively city with a famous University, and the Institute for Theoretical Physics of this University had won a high reputation under the leadership of Professor A. Sommerfeld. This young professor had already achieved great fame. He had published a remarkable book on the theory of the gyroscope, and had presented a very extraordinary paper at the first Solvay Congress in *Brussels in 1911* [French edition at Gauthier-Villars, Paris.1912, p. 316 and p. 403]. In a stroke of genius, he noted that Planck's constant h represented a *quantum of action*, and that the familiar quantum of energy $h\nu$ was only an indirect result of quantizing the action. He made a few curious applications of this revolutionary idea, which P. Langevin immediately used to compute a *magneton*, which differed from the present Bohr magneton only by a factor 2π.

When Bohr's paper on the hydrogen atom was published in 1913, Sommerfeld immediately saw the importance of this new idea. I happened to be in his office when he opened the issue of the *Philosophical Magazine*, which had just arrived; he glanced through it and told me: "There is a most important paper here by N. Bohr, it will mark a date in theoretical physics." And soon after, Sommerfeld started applying his own "quantum of action" method to rebuild a consistent theory of Bohr's atom. This is how the first quantized mechanics was born, and why it progressed so fast. It was definitely Sommerfeld's discovery of the importance of the $\int p\,dq$ integrals that paved the way and these integrals still are at the basis of the whole quantum theory.

Everybody wondered (and still wonders) why the Stockholm committee systematically ignored Sommerfeld's pioneer work in modern physics. Such an omission is actually impossible to understand.

My friend P. P. Ewald gave an excellent summary of Sommerfeld's achievements, and described the life at the Munich Institute for Theoretical Physics, in a Foreword to Volume I of Sommerfeld's lectures ("Mechanics," Academic Press, 1952). The special clarity and the mathematical accuracy of Sommerfeld's lectures were really remarkable. I had the great privilege of attending, as a student, lectures given by some prominent physicists, such as H. A. Lorentz, H. Poincaré, and P. Langevin. But I was especially impressed by Sommerfeld's mastery as a teacher. In his Foreword to Volume I, Ewald quotes a few problems in which Sommerfeld was interested in 1913. Among them is the question of signal velocity in a dispersive medium, a short summary of which is presented in Volume 5, § 22. This was the subject of research suggested to me by Sommerfeld and it resulted in twin papers published by us in the *Annalen der Physik* of 1914. The subject was a fascinating one, but it had, at that time, only academic importance. Experimental verifications were discovered much later, in connection with reflections of radio signals from the Heaviside layers, and also for problems of radar systems. Theoretical applications suddenly appeared with wave mechanics, when Schrödinger discovered that group velocity should be identified with the velocity of particles guided by the waves.

All these modern developments made it advisable to assemble here a systematic presentation of the original papers, which are rather difficult to find nowadays. It is hoped that the present book will be helpful to many readers and save them time and trouble, especially the trouble of recomputing and rediscovering many important features of the general theory.

It is a pleasant duty to thank Dr. E. Erlbach of the Watson Laboratory for preparing translations of the German and French papers.

L. Brillouin

New York
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PREFACE

When a mathematician thinks of wave propagation, he starts by writing a well-known second order differential equation and discussing its peculiar properties. The physicist is interested in these results, but he immediately asks some indiscreet questions about waves in a dispersive medium, when the velocity of propagation is not a constant, but strongly depends upon the frequency. The well-known differential equation is no longer satisfied and must be replaced by a more complicated system of equations, which include the model, the physical mechanism, reacting on the waves and modifying the velocity. Each problem seems different, but nevertheless some general properties may be deduced and some definitions can be found to apply to a wide class of systems.

One of the most important definitions refers to the *group velocity*. It seems to have been first discovered by Lord Rayleigh, who characterized this velocity in sound waves. It is now known to apply to practically all kinds of waves. Let us use the vocabulary of radio engineers and consider a carrier wave, with a superimposed modulation. The *phase velocity* yields the motion of elementary wavelets in the carrier, while the group velocity gives the propagation of the modulation. Lord Rayleigh considered that the group velocity corresponds to the velocity of energy or signals.

This however raised difficulties with the theory of relativity which states that no velocity can be higher than c, the velocity of light in vacuum. Group velocity, as originally defined, became larger than c or even negative within an absorption band. Such a contradiction had to be resolved and was extensively discussed in many meetings about 1910. Sommerfeld stated the problem correctly and proved that no signal velocity could exceed c. I discussed the solution in great detail and gave a complete answer. These original papers and discussions are presented in the first chapters of this book. It was found desirable to reprint completely these papers, which were published during the First World War and are missing in many libraries.

In the following chapters we give a later discussion of the subject, and introduce three different definitions of velocities: A — the *group velocity* of Lord Rayleigh; B — the *signal velocity* of Sommerfeld; C — the *velocity of energy transfer*, which yields the rate of energy flow through a continuous wave and is strongly related to the characteristic impedance.

These three velocities are identical for nonabsorbing media, but they differ considerably in an absorption band.

Some examples are discussed in the last chapter dealing with guided waves, and many other cases of application of these definitions are quoted.

These problems have come again into the foreground, in connection with the propagation of radio signals and radar. Reflection in the Heaviside layers requires a real knowledge of all these different definitions. Group velocity also plays a very important role in wave mechanics and corresponds to the speed of a particle.

The present book should be very useful to physicists and radio engineers and should give them a good basis for new discussions and applications.

L. Brillouin

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CHAPTER I

INTRODUCTION

1. Phase Velocity and Group Velocity

Many modern ideas on wave propagation originated in the famous works of Lord Rayleigh, and the problems we intend to discuss are no exception to this rule. The distinction between phase velocity and group velocity appears very early in Rayleigh's papers.[1] It can be found in his "Theory of Sound"[2] and in many articles reprinted in his "Scientific Papers." The problem is discussed in particular in connection with measurements of the velocity of light;[3] and this is the place where a curious error was introduced regarding the angle of aberration. We shall come back to this point later when discussing a very important paper by P. Ehrenfest (see Section 5 of this chapter).

Let us first remind the reader of the fact that the usual velocity W of waves is defined as giving the phase difference between the vibrations observed at two different points in a free plane wave. It is primarily used for computing interference fringes that make phase differences visible. In a wave

$$\psi = A\cos(\omega t - kx) = A\cos\omega\left(t - \frac{x}{W}\right) \tag{1}$$

we observe the *phase velocity* W

$$W = \frac{\omega}{k} \tag{2}$$

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[1] The very first idea of group velocity appears in a paper by W. R. Hamilton, *Proc. Roy. Irish Acad.* **1**, 267, 341 (1839).

[2] Lord Rayleigh, "Theory of Sound," 2nd ed. (1894). First ed. published, 1877.

[3] Lord Rayleigh, "Scientific Papers," Vol. I, p. 537. 1881.

Another velocity can be defined, if we consider the propagation of a peculiarity (to use Rayleigh's term), that is, of a change in amplitude impressed on a train of waves.

This is what we now call a *modulation* impressed on a *carrier*. The modulation results in the building up of some "groups" of large amplitude (Rayleigh) which move along with the *group velocity U*. In wave mechanics, Schrödinger called these groups "*wave-packets.*" A simple combination of groups obtains when two waves

$$\omega_1 = \omega + \Delta\omega \qquad k_1 = k + \Delta k$$
$$\omega_2 = \omega - \Delta\omega \qquad k_2 = k - \Delta k \tag{3}$$

are superimposed, giving:

$$\begin{aligned}\psi &= A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x)\\ &= 2A\cos(\omega t - kx)\cos(\Delta\omega t - \Delta k x)\end{aligned} \tag{4}$$

This represents a carrier with frequency ω and a modulation with frequency $\Delta\omega$. The wave may be described as a succession of moving beats (or groups, or wave-packets). The carrier's velocity is W [Eq. (2)], while the group velocity is given by U

$$U = \frac{\Delta\omega}{\Delta k} \to \frac{\partial\omega}{\partial k} \quad \text{for} \quad \Delta k \to 0 \tag{5}$$

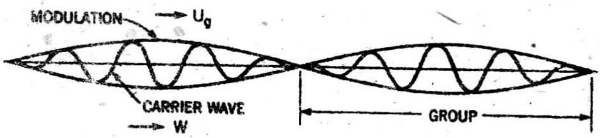

FIG. 1.

The situation is represented in Fig. 1 where we see a succession of wavelets (ω, k) with variable amplitude $(\Delta\omega, \Delta k)$. If we do not pay attention to the detailed motion and observe only the average amplitude distribution, we verify that the amplitude curve moves forward with the group velocity U. Looking more carefully at the detailed vibrations, we may see the wavelets moving within the envelope with their own velocity W. We distinguish two different cases:

(6) $U > W$ The wavelets are building up in front of the group and disappearing in the rear end of the group.

(7) $U < W$ The wavelets are building up at the back end of the group, progressing through the group, and disappearing in the front.

2. Examples and Discussion: Dispersive Media

In a medium where the phase velocity W is a constant and does not depend upon frequency, we have

(8) $$U = W$$

and any kind of signal is propagated without distortion.

More generally, when W is a function of ω (or k), we have

$$U = \frac{\partial \omega}{\partial k}$$

with $\omega = kW$, hence:

(9a) $$U = W + k\frac{\partial W}{\partial k}$$

This is often written with the wave length λ as variable instead of k, when $k = 2\pi/\lambda$; hence,

(9b) $$U = W - \lambda\frac{\partial W}{\partial \lambda}$$

A medium exhibiting a wave velocity $W(k)$ is called a dispersive medium. Vacuum is nondispersive for light ($W = U = c$), but all material media are dispersive. It is impossible to think of a refractive medium without dispersion. The situation is even more complicated, since W depends upon the variables λ (or ω), the density ρ, and the temperature T. In crystals, the direction of propagation is also to be taken into account. We shall restrict our discussions to isotropic media, but we must assume

$$W = W(k,\rho,T) \tag{10}$$

This is where the physicist's viewpoint differs from the mathematician's idealization. Many textbooks on electromagnetic theory discuss material media with

(11)
$\varepsilon \geqslant \varepsilon_0$ dielectric constants of matter and vacuum
$\mu \geqslant \mu_0$ permeabilities of matter and vacuum

but they usually assume ε and μ to be constant, and this is a physical impossibility. The complete problem dealing with the three variables k, ρ T will be examined in Chapter V.

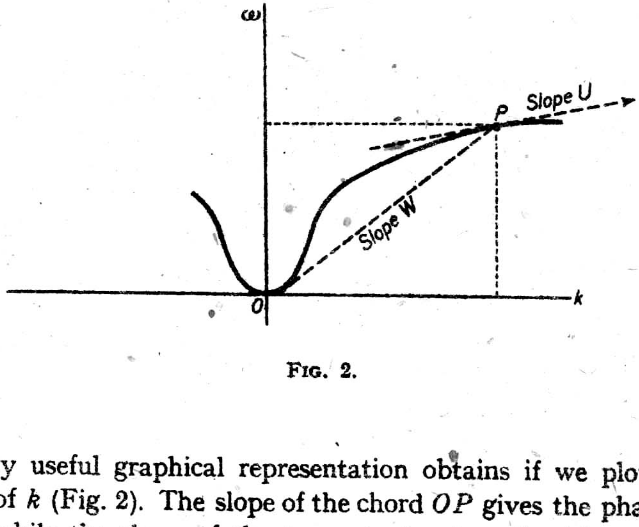

Fig. 2.

A very useful graphical representation obtains if we plot ω as a function of k (Fig. 2). The slope of the chord OP gives the phase velocity W, while the slope of the tangent at point P yields the group velocity U.

The velocity of *light* is a constant in vacuum, but depends upon frequency in material media. The velocity of *sound* is approximately constant for long wavelengths, but depends strongly on the frequency at short wavelengths, especially when the wavelength is of the order of the distance between molecules. Many such examples have been discussed in the literature.[4] The group velocity for sound is then equal to the phase velocity only for long wavelengths.

It was assumed, at the beginning, that the group velocity was actually the velocity at which a finite signal may propagate through the medium, but this is only an approximation. We shall see later that a finite signal is distorted while traveling through the medium, and that its velocity may become very hard to define, on account of the change in the shape.

This is especially true for an absorbing medium. Absorption is strongly frequency dependent, and is always associated with strong dispersion.

As a rule, we shall see that the velocity of a signal does not differ too much from the group velocity, whenever absorption and dispersion are small. Otherwise, the velocities may differ widely.

Let us now discuss a few interesting examples, to which the reader may add a great variety of problems discussed by L. Brillouin in a previous book.[4]

Rayleigh discusses[5] the problem of wave propagation along a bar, and obtains an equation for lateral vibrations:

$$\frac{\partial^2 y}{\partial t^2} + K^2 b^2 \frac{\partial^4 y}{\partial x^4} = 0 \tag{12}$$

This propagation is frequency dependent, and for a wavelength λ one obtains a velocity

$$W = \frac{2\pi Kb}{\lambda} = Kbk \tag{13}$$

with $k = 2\pi/\lambda$.

In this example, Rayleigh discusses the problem of group velocity. He assumes, more generally,

$$W = B\lambda^n = B'k^{-n} \tag{14}$$

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[4] See, for instance: L. Brillouin, "Wave Propagation in Periodic Structures." McGraw-Hill, New York, 1946. Reprinted, Dover, New York, 1953. L. Brillouin and M. Parodi, "Propagation des ondes dans les milieux périodiques." Masson, Paris, 1956.

[5] Lord Rayleigh, reference 2, Vol. I, Section 191, p. 301.

which results, by our formulas (8) or (9), in

$$U = W(1 - n) \tag{15}$$

For lateral vibrations of *bars*,

$$n = -1 \qquad U = 2W \tag{16}$$

The group velocity is thus twice as large as the phase velocity. This is a typical example of case (7) in Section 1 above.

In another chapter of "Theory of Sound,"[6] Rayleigh discusses surface waves on water. Assuming a density ρ, a depth l, gravity g, and surface tension T, he obtains the general formula for the phase velocity[7]

$$W^2 = \frac{g}{k} + \frac{Tk}{\rho} \tanh(kl) \tag{17}$$

a formula exhibiting a strong dependence on k.

In many important cases, the depth l can be considered as practically infinite (deep water waves); thus the hyperbolic tangent is 1, and hence

$$W^2 = \frac{g}{k} + \frac{Tk}{\rho} \qquad k = \frac{2\pi}{\lambda} \tag{18}$$

When λ is great, k is small, and the waves move mainly under gravity, with a velocity

$$W = \left(\frac{g}{k}\right)^{1/2} \quad \text{when} \quad k^2 \ll \frac{g\rho}{T}, \quad \lambda^2 \gg \frac{4\pi^2 T}{g\rho} \tag{19}$$

This is the case of long waves on deep sea. For small ripples, k is large, the second term in Eq. (18) is dominant, and

$$W = \left(\frac{Tk}{\rho}\right)^{1/2} \tag{20}$$

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[6] Lord Rayleigh, reference 2, Vol. II, Chapter XX.

[7] Lord Rayleigh, reference 2, Vol. II, p. 344, Eq. (7).

Between these extreme cases, there is a minimum velocity W_0 corresponding to λ_0 and τ_0 values for wavelength and period, respectively.

$$(21)\qquad W_0 = \left(\frac{4Tg}{\rho}\right)^{1/4} \qquad \lambda_0 = 2\pi\left(\frac{T}{g\rho}\right)^{1/2} \qquad \tau_0 = 2\pi\left(\frac{T}{4g^3\rho}\right)^{1/4}$$

According to Eq. (19), *long waves on deep sea* yield a power of $n = \frac{1}{2}$ and hence a group velocity

$$(22)\qquad U = \frac{1}{2}W$$

according to Eq. (15). This is a typical example of case (6) in Section 1 above.

Short ripples moving under *surface tension,* on the contrary, correspond to $n = -\frac{1}{2}$ in Eq. (20); hence

$$(23)\qquad U = \frac{3}{2}W$$

which is an example of case (7).

A very simple experiment can easily be made and provides an excellent example of group velocity. Just throw a stone in a pond, and look at the "rings" produced on the surface. They are composed of a small number of short ripples. The system as a whole propagates with the group velocity U but each individual ripple moves with the phase velocity W. Since $W < U$, these ripples are building up along the outside ring, moving more slowly than the ring, and disappearing on the inside of the ring.

3. Groups and Signals

The preceding example may serve as an introduction to the discussion of signals. *Groups* were defined by Rayleigh as moving beats [Eqs. (4) and (5)] following each other in a regular pattern. A *signal* is a short isolated succession of wavelets, with the system at rest before the signal arrived and also after it has passed. A signal may be sharply defined in time and duration, in which case its frequency spectrum extends from $-\infty$ to $+\infty$, or it may have a finite spectrum, and exhibit no absolutely sharp boundaries. These problems were extensively discussed elsewhere.[8]

We shall assume a signal carried by a *carrier-frequency* ω_0 and characterized by a *modulation* curve $C(t)$. The complete signal sent along the line at the input $x = 0$ is

$$C_1(t,0) = C(t) \cos \omega_0 t \tag{24}$$

Let us now analyze the *modulation* $C(t)$ in a Fourier integral, assuming that this modulation has a finite spectrum extending from 0 to ω_m:

$$C(t) = \int_{\omega=0}^{\omega_m} B_\omega \cos(\omega t + \phi_\omega)\, d\omega \tag{25}$$

where B_ω is the amplitude and ϕ_ω the phase of the ω component. The input signal [Eq. (24)] is represented by the Fourier integral

$$\begin{aligned} C_1(t,0) &= \int_{\omega=0}^{\omega_m} B_\omega \cos(\omega t + \phi_\omega) \cos(\omega_0 t)\, d\omega \\ &= \frac{1}{2}\int_{\omega=0}^{\omega_m} B_\omega \{\cos[(\omega_0 + \omega)t + \phi_\omega] + \cos[(\omega_0 - \omega)t - \phi_0]\}\, d\omega \end{aligned} \tag{26}$$

The resulting spectrum now extends from $(\omega_0 - \omega_m)$ to $(\omega_0 + \omega_m)$ and thus covers a band $2\omega_m$. For simplicity's sake, we may assume

$$\omega_0 \geqslant \omega_m \tag{27}$$

and avoid negative frequencies. The line along which propagation occurs is characterized by a certain relation between ω and k, as visualized in Fig. 2.

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[8] L. Brillouin, reference 4, Chapter V, p. 78. L. Brillouin and M. Parodi, reference 4, Chapter V, p. 81. L. Brillouin, "Science and Information Theory," Chapter 8, p. 86. Academic Press, New York, 1956.