G. Scharf

Finite Quantum Electrodynamics

The Causal Approach

Second Edition

有限量子电动力学

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Preface

Quantum field theory as it is usually formulated is full of problems with ultraviolet and infrared divergences. This is somewhat surprising, because there is a simple way out which one learns in mathematics. One must only adopt the following two rules. First, use well-defined quantities only, for example free fields. Second, make justified operations only in the calculations; in particular do not multiply certain distributions by discontinuous step functions. If one really follows these rules, then no infinity can appear and life is beautiful. The question then is how to construct the standard theory according to these rules. This one can learn from an old paper by Epstein and Glaser (*Annales de l'Institut Poincaré A 19, p. 211 (1973)*). The main tool in this method is causality.

The causal method was developed by Stückelberg and Bogoliubov in the 1950s. One reason for the limited resonance it found was perhaps the highly non-trivial nature of the causality condition. We therefore start slowly. After a chapter on the classical Dirac theory of electrons and positrons and the quantization of free fields, we study the external field problem in some detail. We will find that the (second quantized) scattering matrix (S-matrix) for this problem is uniquely determined up to a phase. This phase contains physical effects, namely the so-called vacuum polarization which is produced by the external field. Therefore, it is needed to complete the construction of the S-matrix, and here is the place where the causality condition comes in for the first time. With this experience we are then able to construct the S-matrix of full QED by causal perturbation theory in Chap. 3. The important point is that this directly leads to the finite ("renormalized") perturbation series. In fact, no divergent Feynman integral and no ultraviolet cutoff will appear in this book, explaining why the title "Finite QED" was chosen.

It is a common belief that QED with a cutoff or scale parameter should be considered as part of a more fundamental theory where the scale parameter disappears, and that the theory is only mathematically well defined in this bigger framework. We will see that there is no scale parameter in QED in the causal approach if the electrons are massive. If one considers massless fermions, then a scale parameter appears in a natural way, because the central splitting solution (Sect. 3.2) no longer exists. This suggests that it seems indeed necessary to study a bigger theory if one wants to attack the mass problem. But if we take the electron mass as a given finite parameter, QED still has a good chance of being well defined. In fact, the perfect agreement of the perturbative results with experiment cannot be an accident; there must exist well-defined objects (perhaps the adiabatically switched S-matrix $S(g)$ of Sect. 3.1) which are approximated by perturbation theory.

The fact that the causal theory is perturbative has not only a technical but also a deep physical reason. In any realistic quantum field theory one must draw a sharp distinction between the fundamental fields that appear in the elementary interaction and the asymptotic states describing the real incoming and outgoing particles. This is well known today from the theory of strong interactions (quantum chromodynamics, QCD) where the quark fields are the fundamental Fermi fields, while the mesons and nucleons are complicated bound states of them. But even the electron is complicated because it carries the Coulomb field, so it must be regarded as a bound state where (scalar) photons are confined to a Dirac field. Compositeness is the normal case. Only the photon and the neutrini seem to be elementary in the sense that they can simply be generated by fundamental fields. In the causal theory the very hard problem of the asymptotic states is clearly separated from the rest of the theory by the method of adiabatic switching: the interaction is multiplied by a test function $g(x)$ and one performs the adiabatic limit $g \to 1$ at the very end in observable quantities. This means that the confinement is switched off in the asymptotic region in a "gedanken-experiment", so that free fields are coming out, instead of the complicated real physical particles. The switching is then removed in the adiabatic limit. From the study of this limit one can learn something about the structure of the real asymptotic states. It turns out that the limit does not always exist. Only if the right inclusive cross sections are considered does the limit come out finite and unique. In this way the S-matrix itself dictates the structure of the physical particles, as it must be. This highly important fact, which is even more important in non-abelian theories, can already be seen in perturbation theory, as we will discuss at the end of Chap. 3. But it seems to be rather hopeless to jump by some non-perturbative guess directly to the correct description of the asymptotic states.

The inductive construction of the S-matrix enables us to give simple inductive proofs of the various properties of the theory, in particular gauge invariance and unitarity. These themes are described in Chap. 4. The discussion of other electromagnetic couplings in Chap. 5. brings in new features which are important for preparing the extension of the causal method to non-abelean gauge theories. One might regret that this subject is not yet included, but the Epilogue gives a short account of the present status of this field. In the historical introduction the various lines of development in quantum field theory are discussed. From the beginning in the 1920s until today this was a fascinating sequence of successes and failures, where each attempt contained its piece of truth.

The book differs considerably from its first edition: Chapter 3 was completely rewritten and the Chaps. 4 and 5 are new. The bibliographical notes give some hints for further reading.

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Zürich, May 1995 *G. Scharf*

Contents

0. Preliminaries

We start the numbering with zero because this chapter is preparatory. At the beginning of each chapter we want to make some general introductory remarks because, we think, the reader has a right to know in advance why the material that follows is presented to him. We begin with an introduction into the history of quantum field theory. To understand the striking success of this theory, it is helpful and clarifying to remember how the fundamental ideas have been introduced in the past and how they got modified in the course of time. After this historical introduction of those concepts we start with their physical introduction.

The object of physics is the description of observable phenomena in space and time and the investigation of the mathematical structure behind these phenomena. Therefore in the first section the 4-dimensional space of space-time points and the corresponding transformation group of the reference systems is described. The tensor calculus, which is briefly discussed in Sect. 0.2, is a tool to write the equations in a form independent of the reference system. The third section is concerned with some basic concepts of scattering theory. As we shall see much later, it is difficult, in general, to formulate the time-evolution of a system in quantum field theory, contrary to non-relativistic quantum mechanics. In this situation, scattering theory becomes of central importance. We show how the scattering matrix can be constructed using causality instead of dynamical equations. This is precisely what we will do in the case of full QED in Chap. 3. Causality will be the cornerstone in the book.

0.0 Historical Introduction

The dawn of quantum field theory coincides with the development of quantum mechanics in the 1920's. When M. Born and P. Jordan (*Zeitschrift f. Physik 34, 886 (1925)*) clarified the structure of Heisenberg's matrix mechanics, they added a chapter IV with the title "Remarks on Electrodynamics". They pointed out that the quantum mechanical treatment of the harmonic oscillator, which was of crucial importance for the discovery of the theory, is also relevant for the electromagnetic field: Although the latter is a system of infinitely many degrees of freedom, the theory of the one-dimensional oscillator is sufficient for its treatment, because the radiation field can be regarded as a system of uncoupled oscillators. Then the electric and magnetic field strength $\boldsymbol{E}, \boldsymbol{H}$ with periodic time dependence become matrices. The authors, therefore, used the notion "matrix electrodynamics". But they only considered the free electromagnetic field.

The name quantum electrodynamics (QED) was introduced by P.A.M. Dirac (*Proc. Roy. Soc. London A 114, 243 (1927)*) in his paper on "The Quantum Theory of Emission and Absorption of Radiation" after Schrödinger's formulation of quantum mechanics in 1926. Dirac had the time-dependent perturbation theory at his disposal, therefore, he was able to treat the radiation field in interaction with an atom. He observed that light quanta must obey Bose-Einstein statistics and calculated Einstein's A- and B-coefficients for the emission and absorption rates. Here spontaneous emission was explained for the first time. The procedure of quantizing the radiation field still remained somewhat unclear. This point was further considered by P. Jordan and W. Pauli (*Z. Phys. 47, 151 (1928)*) in their paper "On Quantum Electrodynamics of Fields without Charges". They gave a Lorentz invariant quantization of the electromagnetic field and introduced the invariant D-function which was later called Jordan-Pauli distribution. They arrived directly at the commutation relations for the electric and magnetic fields $\boldsymbol{E}, \boldsymbol{H}$ and noticed that there exist no simple invariant commutation relation for the vector potential. They also noticed the difficulty of the infinite zero-point energy. Jordan continued this line of research together with E. Wigner (*Z. Phys. 47, 631 (1928)*) in the paper "On Pauli's Exclusion Principle", where they showed that Pauli's principle implies field quantization with anticommutators. This led them to an elegant theory of the Fermi gas.

At the same time Dirac established the second pillar of QED, namely the relativistic equation for the electron in his paper "The Quantum Theory of the Electron" (*Proc. Roy. Soc. London 117, 610 and 118, 351 (1928)*). This famous equation immediately explained the spin of the electron and its magnetic moment $e\hbar/2mc$, as well as the fine-structure of the spectrum of the hydrogen atom. Despite these brilliant successes, there was a serious difficulty in the theory which was realized by Dirac: The equation has solutions with unbounded negative energy. This problem occupied Dirac for almost two years. At the beginning of 1930 (*Proc. Roy. Soc. London 126, 360 (1930)*) he gave a solution in his paper "A Theory of Electrons and Protons" (originally he thought the negative energy states to be protons). He interpreted the theory as a multiparticle theory and used the exclusion principle for the electrons. He did not put all pieces together, because he was not using Jordan and Wigner's method for quantization of Fermi fields which would be the appropriate tool, but developed a picture of his own in his "hole theory". It rests on the assumption that all states with negative energy are filled up with electrons, so that no electron can jump into one of these occupied states according to the exclusion principle. This new picture of the vacuum state has observable consequences in electron-photon scattering, and it predicts new effects: A hole in the sea of negative states appears as a particle with opposite (positive) charge. Dirac first thought that this must be the proton, because no other particle with positive charge was known. But then the two particles would annihilate in a hydrogen atom. Finally (*Proc. Roy. Soc. London 133, 60 (1931)*) he assumed that the holes are new, yet unknown "anti-electrons" with the same mass as electrons but charge $+e$. By analogy he also thought that anti-protons might exist. When the anti-electron (positron) was indeed found by C.D. Anderson in the cosmic rays in 1932, this was the first particle correctly predicted by theory. The anti-proton was observed much later in 1955.

As already said, Dirac with his hole theory did not follow the ideas of quantum field theory. This direction was further pursued by W. Heisenberg and W. Pauli in their paper "On Quantum Dynamics of Wave Fields" (*Z. Phys. 56, 1 (1929)*). Here the general method of canonical quantization was systematically developed. The problem was reduced to quantum mechanics by dividing the 3-space into cells and treating the field variables in these cells like the mechanical coordinates and momenta. Pauli has sometimes used this old method in later years for basic reasoning. When the method was applied to electrodynamics, some difficulties appeared, because the time-component of the vector potential has no conjugate momentum. This problem was brilliantly circumvented by introducing a gauge-fixing term, as we call it today. However, for the electron field satisfying the Dirac equation the two possibilities with commutation or anticommutation relations were treated upon the same footing. Obviously, the connection of spin and statistics was not yet understood. For Pauli this was a theme for a long time (*Phys. Rev. 58, 716 (1940)*). The problem of the negative energy states was still not solved, as well as the zero-point energy of the radiation field and the infinite self-energy of the electron.

That the zero-point energy of the electromagnetic field in infinite space has no physical meaning was clear to many authors. But the radiation field poses more problems. To treat the interaction with matter, it is necessary to use potentials. Then, however, it is difficult to perform the quantization in a manifest Lorentz covariant form. Dirac in the second edition of his book on "Quantum Mechanics" (*Oxford 1935*) gave an elegant solution to the problem using results of E. Fermi (*Rev. Mod. Phys. 4, 125 (1932)*). The positron problem was even harder because there is a polarization of the vacuum (*W. Heisenberg, Z. Phys. 90, 209 (1934)*). Heisenberg found a pragmatic solution: he quantized the free electron-positron field in accordance with Dirac's hole theory and then developed perturbation theory. At the same time W.H. Furry and J.R. Oppenheimer wrote a paper "On the Theory of Electron and Positive" (*Phys. Rev. 45, 245 (1934)*) where they discuss (second) quantization of the Dirac field in the modern way. When Pauli summarized the status of the theory in his review article "Relativistic Field Theory of Elementary Particles" (*Rev. Mod. Phys. 13, 203 (1941)*), he quantized all interesting fields in a completely satisfactory manner, apart from a small reservation in case of the Dirac field. This article was called the "New Testament" by the younger collaborators of Pauli in contrast to his work of 1933 on quantum mechanics (*Handbuch der Physik, 2. Aufl., Bd. 24/1*), which was the "Old Testament".

However, the situation with respect to the other infinities that are due to interaction could not be improved until after the Second World War. The key point was to formulate QED in a manifest relativistically covariant form. This was independently achieved by S. Tomonaga and collaborators, J. Schwinger and R.P. Feynman in different manners. They won the Nobel prize together in 1965. Tomonaga's work (*Progr. Theoret. Phys. Kyoto 2, 101 (1947)*) was closest to the older quantum field theory, because he started from the Schrödinger picture, went over to the Heisenberg picture and established perturbation theory. Schwinger (*Phys. Rev. 74, 1439 (1948), 75, 651 (1949)*) worked in the intermediate interaction representation which Tomonaga had implicitly also used, and constructed the Lorentz invariant collision operator (S-matrix). He calculated mostly in x-space which required ingenious formal tricks, because most objects are much more singular here than in momentum space. Feynman worked in a totally different way. In his paper "Space-Time Structure of Quantum Electro Dynamics" (*Phys. Rev. 76, 769 (1949)*) he avoided quantized fields altogether, using a quantum mechanical propagator theory instead. But the field quantization is hidden in the rules for many-body processes and in the choice of the propagator functions. F.J. Dyson (*Phys. Rev. 75, 486 (1949)*) showed the equivalence of this theory with Tomonaga's and Schwinger's and derived the Feynman rules by means of of quantum field theory. Feynman's formulation in momentum space was of greatest importance for the further development of field theory and particle physics, because it gives by far the simplest scheme for the explicit calculations.

Unfortunately, the Feynman rules still lead to ill-defined integrals which are ultraviolet and partially also infrared divergent. But in the covariant theory it was possible to calculate unique finite results which are in perfect agreement with experiments. This was achieved by regularization of the integrals and absorption of the infinities into the mass and charge terms, the well-known method of renormalization (F.J. Dyson *Phys. Rev. 75, 1736 (1949)*). Although the final results of the theory were certainly correct, it was clear that this was not yet the right formulation. Tomonaga said in his Nobel lecture: "It is a very pleasant thing that no divergence is involved in the theory except for the two infinities of electronic mass and charge. We cannot say that we have no divergences in the theory, since the mass and charge are in fact infinite." And Feynman in his Nobel lecture (*Science 153, 699 (1966)*) was even more critical of his own work: "I think that the renormalization theory is simply a way to sweep the difficulties of the divergences of electrodynamics under the rug. I am, of course, not sure of that." Twenty years later in his popular book with the remarkable title "The Strange Theory of Light and Matter" (*Princeton N.J. 1985*) he still wrote: "What is certain is that we do not have a good mathematical way to describe the theory of quantum electrodynamics." Another critic was Dirac. He called the theory "an ugly and incomplete one" (*Proc. Roy. Soc. A 209, 291 (1951)*). In his book "Dreams of a Final Theory" (*London 1993, p. 91*) S. Weinberg reported on discussions with Dirac and wrote: "I did not see what was so terrible about an infinity in the bare mass and charge as long as the final answers for physical quantities turn out to be finite and unambiguous and in agreement with experiment. It seemed to me that a theory that is as spectacularly successful as quantum electrodynamics has to be more or less correct, although we may not be formulating it in just the right way. But Dirac was unmoved by these arguments. I do not agree with his attitude towards quantum electrodynamics, but I do not think that he was just being stubborn; the demand for a completely finite theory is similar to a host of other aesthetic judgements that theoretical physicists always need to make." Dirac's point, perhaps, was that mathematical consistency is more fundamental than aesthetic judgements.

The third Nobel laureate of 1965 said nothing about the divergence problems, instead Schwinger made the following introductory remark: "I shall begin by describing to you the logical foundations of relativistic quantum field theory. No dry recital of lifeless "axioms" is intended ..." What are the lifeless axioms? In the 1950's A.S. Wightman and others (*R.F. Streater and A.S. Wightman, PCT, Spin and Statistics, and All That, New York 1964*) started to analyse the general structure which underlies all quantum field theories. From the well understood theory of free fields they extracted general properties (formulated as axioms) and studied the relations between them with rigorous mathematical methods. The resulting "general theory of quantized fields" (this better name is the title of a book by *R. Jost, Providence, Rhode Island 1965*) supplied various important results. But the main question whether the basic notions apply to realistic theories remained open. Only in lower dimensions non-trivial models satisfying the Wightman axioms have been constructed (*J. Glimm, A. Jaffe, Quantum Physics, Springer-Verlag 1981*). The failure of some constructive methods in four dimensions has given rise to speculations that a non-perturbative definition of QED might not exist. One must be careful with such statements, because one can only prove that *a particular construction* does not work.

There exists another more pragmatic approach which is the basis of this book. It goes back to Heisenberg (*Z. Phys. 120, 513 (1943)*) and takes the scattering operator (S-matrix) as the basic quantity. The S-matrix maps the asymptotically incoming, free fields on the outgoing ones and, hence, it should be possible to express it completely by the well-defined free fields. E.C.G. Stückelberg and collaborators (*Helv. Phys. Acta 23, 215 (1950), 24, 153 (1951)*) showed that this is possible in perturbation theory if one uses a causality condition in addition to unitarity of the S-matrix. Later on N.N. Bogoliubov and D.V. Shirkov (*Introduction to the Theory of Quantized Fields, New York 1959*) simplified the causality condition by using the important tool of adiabatic switching with a test function. This tool must be used for mathematical reasons because the S-matrix is an operator-valued functional and not an operator, and also for physical reasons since the real asymptotic states are not simply generated by free fields, as briefly discussed in the preface.

Unfortunately, these authors did not solve the divergence problems because they arrived at the usual defective expression for the S-matrix involving naively defined time-ordered products. As mentioned in the preface, the program was successfully carried through for scalar theories by H. Epstein and V. Glaser in 1973 (*Annales de l'Institut Poincaré A 19, 211 (1973)*). In their method the perturbation series for the S-matrix was constructed inductively, order by order, by means of causality and translation invariance; unitarity was not used. The most delicate step in this construction is the decomposition of distributions with causal support into retarded and advanced parts. If this distribution splitting is carried out without care by multiplication with step functions, then the usual ultraviolet divergences appear. But if it is carefully done by first multiplying with a C^∞ function and then performing the limit to the step function, everything is finite and well-defined. In this way the ultraviolet problem which has plagued field theorists for more than fifty years does not arise at all. Unfortunately, it is still not clear how the perturbation series can be summed up. Therefore, problems occurring in partial resummation, like the Landau pole (*M. Gell-Mann, F. Low, Phys. Rev. 95, 1300 (1954)*), cannot be treated yet. One should notice that this problem does not arise, if one considers the adiabatically switched S-matrix $S(g)$ (Sect. 3.1).

Summing up, we have looked at the history of quantum electrodynamics like a doctor examining the course of a disease. In fact, the force driving this history was mainly the attempt to "cure" the illness of the various divergences. The infinities were present in QED from the very beginning and their slow disappearance indicates our progress in understanding. Sometimes the disease has been considered so grave that radical treatment was recommended. But until now quantum field theory has always survived and we hope that it will be completely healthy one fine day.

0.1 Minkowski Space and the Lorentz Group

The framework of a physical description is the four-dimensional real space $\mathbb{R}^4$ of space-time points $x = (x^0, x^1, x^2, x^3) = (x^\mu), x^0 = ct$. The velocity of light c has been introduced into the time component in order to have the same dimension in all four components of x. Throughout we use the convention that greek indices assume the values 0,1,2,3, whereas latin indices are used for the spatial values 1,2,3. Specifying the position x of a physical object as a function of time t, defines a curve in $\mathbb{R}^4$. The light rays outgoing from the origin move on the light-cone

$$c^2t^2 - |\boldsymbol{x}|^2 = 0. \tag{0.1.1}$$

This double-cone consists of the past-cone $t < 0$ and the future-cone $t > 0$. A change of the frame of reference is described by a linear transformation

$$x \longrightarrow x' = \Lambda x, \tag{0.1.2}$$

where Λ is a real 4×4-matrix. Introducing components with respect to a basis $e_\mu, \mu = 0, 1, 2, 3$

$$x = x^\mu e_\mu,$$

the transformation (0.1.2) is written as follows

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{0.1.3}$$

where the convention of summing over double upper and lower indices is always assumed. The reason for using upper and lower indices will be explained in the following section.

The basis of relativity is the principle of constant velocity of light. In view of (0.1.1) it can be expressed as follows: If

$$(x^0)^2 - \boldsymbol{x}^2 = 0$$

in one frame of reference then this also holds in another frame

$$(x'^0)^2 - \boldsymbol{x}'^2 = 0.$$

It is convenient to write the quadratic forms appearing here as

$$Q(x) = x^T g\, x \tag{0.1.4}$$

$$Q'(x) = Q(\Lambda x) = x^T \Lambda^T g\, \Lambda\, x, \tag{0.1.5}$$

where

$$g = (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{0.1.6}$$

is the fundamental metric tensor. Both forms (0.1.4, 5) vanish for fixed $\boldsymbol{x}$ if $x^0 = \pm|\boldsymbol{x}|$, therefore

$$Q'(x) = \lambda(x^0 - |\boldsymbol{x}|)(x^0 + |\boldsymbol{x}|) = \lambda((x^0)^2 - \boldsymbol{x}^2) = \lambda\, Q(x).$$

The case $\lambda \neq 1$ corresponds to a change of units which we disregard. Then we arrive at

$$x^T \Lambda^T g\, \Lambda\, x = x^T g\, x$$

for all $x \in \mathbb{R}^4$, or

$$\Lambda^T g\, \Lambda = g. \tag{0.1.7}$$

We emphasize that we have used the condition of constant $x^2 = x'^2$ only for light rays ($x^2 = 0$). All transformations satisfying (0.1.7) are called Lorentz transformations. They obviously form a group, the Lorentz group $\mathcal{L}$. Equation (0.1.7) suggests the introduction of the indefinite scalar product

$$(x,y) = x^T g\, y = x^0y^0 - \boldsymbol{x}\cdot\boldsymbol{y} = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3. \qquad (0.1.8)$$

It is invariant under Lorentz transformations

$$(x', y') = (\Lambda x, \Lambda y) = (\Lambda x)^T g\, \Lambda y = x^T \Lambda^T g\, \Lambda y = x^T g\, y = (x,y).$$

The four-dimensional real vector space with scalar product (0.1.8) is called Minkowski space $\mathbb{M}$. Lorentz transformations are the congruency transformations of $\mathbb{M}$. The elements of $\mathbb{M}$ are called points or (four) vectors in the following.

There are three classes of vectors in $\mathbb{M}$: (i) time-like vectors x with $x^2 > 0$, (ii) space-like vectors y with $y^2 < 0$ and (iii) light-like vectors z with $z^2 = 0$. Each class is mapped into itself under Lorentz transformations because x^2 remains constant. We shall often find that functions of a four-vector x behave differently for time-like or space-like x. A three-dimensional surface S in $\mathbb{M}$ is called time-like or space-like if any tangent vector to S is time-like or space-like, respectively. Two disjoint sets X, Y of points are space-like separated if every vector $x - y, x \in X, y \in Y$ is space-like. Then it is impossible to connect the points x, y in a causal way, for instants by light signals. If $x - y$ is time-like, then the two points are causally connected. This causal structure of Minkowski space will be of crucial importance later.

Equation (0.1.7) implies det $\Lambda = \pm 1$ for all $\Lambda \in \mathcal{L}$. Examples of determinant $= -1$ are time-reflection T and space-reflection P (parity transformation)

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \qquad (0.1.9)$$

The Lorentz transformations Λ with det $\Lambda = +1$ form the subgroup

$$\mathcal{L}_+ = SO(1,3)$$

of $\mathcal{L}$. It is a special pseudo-orthogonal group. The defining Eq. (0.1.7) means that the rows and columns of a Lorentz matrix $\Lambda^\mu{}_\nu$ are orthogonal with respect to the Minkowski scalar product (0.1.8), for example

$$\Lambda^0{}_\mu \Lambda^0{}_\nu - \sum_{j=1}^{3} \Lambda^j{}_\mu \Lambda^j{}_\nu = \begin{cases} 0, & \text{for } \mu \neq \nu \\ 1, & \text{for } \mu = \nu = 0\,. \\ -1 & \text{for } \mu = \nu \neq 0 \end{cases} \qquad (0.1.10)$$

Taking $\mu = \nu = 0$,we have

$$(\Lambda^0{}_0)^2 - \sum_{j=1}^{3} (\Lambda^j{}_0)^2 = 1$$