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（影印版）47

V. P. Khavin N. K. Nikol′skij (Eds.)

Commutative Harmonic Analysis I

General Survey, Classical Aspects

交换调和分析 I

总论，古典问题

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Preface

This volume is the first in the series devoted to the commutative harmonic analysis, a fundamental part of the contemporary mathematics. The fundamental nature of this subject, however, has been determined so long ago, that unlike in other volumes of this publication, we have to start with simple notions which have been in constant use in mathematics and physics. Planning the series as a whole, we have assumed that harmonic analysis is based on a small number of axioms, simply and clearly formulated in terms of group theory which illustrate its sources of ideas. However, our subject cannot be completely reduced to those axioms. This part of mathematics is so well developed and has so many different sides to it that no abstract scheme is able to cover its immense concreteness completely. In particular, it relates to an enormous stock of facts accumulated by the classical "trigonometric" harmonic analysis. Moreover, subjected to a general mathematical tendency of integration and diffusion of conventional intersubject borders, harmonic analysis, in its modern form, more and more rests on non-translation invariant constructions. For example, one of the most significant achievements of latter decades, which has substantially changed the whole shape of harmonic analysis, is the penetration in this subject of subtle techniques of singular integral operators. On the other hand, the traditional topics, such as studies of convolution equations, spectral theory of functions and ideals of convolution algebras, methods of theory of analytic functions in harmonic analysis on semigroups, etc., also occupy an important place in other surveys of this series as well as in harmonic analysis itself.

Below we list some of the topics which, we hope, will be covered in this series.

1. Methods and structure of commutative harmonic analysis. This article is included in the first volume and devoted to the foundations of harmonic analysis, a brief outline of its history, structure and connections with other subjects.
2. Classical themes in Fourier analysis. This is, in some sense, a guide to "trigonometric" Fourier analysis, where new achievements are given together with the results already included in a famous book of Zygmund $[89]_K$.*
3. Methods of singular integrals. Harmonic analysis in $\mathbb{R}^n$.
4. Multiple Fourier series and integrals.
5. Group-theoretic methods of commutative harmonic analysis.
6. Convolution equations and analysis of classical groups of translations.

* $[\cdot]_K$ denotes a corresponding reference in S.V. Kislyakov's article.

7. Analysis of classical semigroups.
8. Tauberian theorems in harmonic analysis.
9. The uncertainty principle in harmonic analysis.
10. Probabilistic methods in harmonic analysis.
11. Exceptional sets.
12. Harmonic analysis in physics.

V.P. Khavin N.K. Nikol'skij

List of Editors, Contributors and Translators

Editor-in-Chief

R.V. Gamkrelidze, Academy of Sciences of the USSR, Steklov Mathematical Institute, ul. Vavilova 42, 117966 Moscow, Institute for Scientific Information (VINITI), ul. Usievicha 20 a, 125219 Moscow, USSR

Consulting Editors

V.P. Khavin, Leningrad University, Department of Mathematics, Staryj Peterhof, 198094 Leningrad, USSR

N.K. Nikol'skij, Steklov Mathematical Institute, Fontanka 27, 191011 Leningrad, USSR

Contributors

E.M. Dyn'kin, Leningrad Institute of Electrical Engineering, Department of Mathematics, Prof. Popov Street 5, Leningrad, USSR

V.P. Khavin, Leningrad University, Department of Mathematics, Staryj Peterhof, 198094 Leningrad, USSR

S.V. Kislyakov, Steklov Mathematical Institute, Fontanka 27, 191011 Leningrad, USSR

Translators

D. Khavinson, The University of Arkansas, Department of Mathematical Sciences, Fayetteville, AR 72701, USA

S.V. Kislyakov, Steklov Mathematical Institute, Fontanka 27, 191011 Leningrad, USSR

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I. Methods and Structure of Commutative Harmonic Analysis

V.P. Khavin

Translated from the Russian
by D. Khavinson

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Introduction

Many really significant "final" scientific achievements share the following two characteristics. First of all, they are sufficiently trivial, and, hence, can become of "common use", i.e., necessary and conventional. Yet, by "conventional" here we, by no means, understand "lying on a surface." Achievements, we are talking about, become trivial as a result of the development, which is itself far from trivial and, sometimes, even painful, and also, as a result of a frequent and wide use.

Secondly, they are sufficiently vague and hence admit many useful refinements and interpretations.

"A more or less arbitrary function is linear in the small." This fairly well-known truth has been obtained as a result of a long and complicated search. Its immense concretizations, refinements and applications form what is called "differential calculus" and an enormous number of people is bound to master it nowadays.

Harmonic analysis can also be surpressed into one short statement, conventional and vague. Here it is.

(*) Every function is a sum of harmonic oscillations ("harmonics").

A notion of "harmonics" is associated with the specific group of transformations. For example, if a question concerns functions defined on the real axis $\mathbb{R}$, which is interpreted as a group with respect to the usual addition, their "harmonics" are functions $ce^{i\lambda x}$ and (*) can be refined as follows.

(**) For every function $f(x)$ defined on the real axis $\mathbb{R}$, there is a function $\hat{f}(\lambda)$ such that $f(x)$ coincides with the sum of all "harmonics" $\hat{f}(\lambda)e^{i\lambda x}$ (over all $\lambda \in \mathbb{R}$). The correspondence $f \to \hat{f}$ is one-to-one, so $\hat{f}$ contains all the information about f. Moreover, in many situations it is more convenient to extract this information from $\hat{f}$ rather than from f.

As mathematical statements, (*) and (**) do not withstand criticism. However, it is precisely because of their vagueness, they are generally valid and important. Although not really correct or even intelligible from the mathematical viewpoint, those statements admit numerous (and rather precise) interpretations and far reaching generalizations which already withstand testing by means of mathematics, physics and technology. The principle of natural science which those statements convey, judged by the critieria of universality of applications and influence on physical and mathematical thinking, can be compared to differential and integral calculus. Problems of harmonic analysis, formulated within one paragraph and without any preparations (in particular, not yet specifying why and in what situations it is more convenient to work with $\hat{f}$ rather than f) consist of clarifying the exact sense of this principle, searching for its generalizations, consequences and applications. Harmonic analysis investigates a certain general "corpuscular-wave dualism" as behind each object described by the function f of one or several real variables (and also much more general objects) there stands "wave" ("spectral", "oscillatory",...) picture described by the function $\hat{f}$, not always visible but inseparable from that object.

Ideas, methods and results of harmonic analysis extend from theory of differential equations to group theory, from arithmetic to probability theory. It serves as the language in many parts of modern physics and technology. The function $\hat{f}$ can be literally seen (e.g., by looking at some diffraction pictures) and heard (e.g., in Helmholtz' acoustic analyzer). It also appears in theory of communications, quantum mechanics, theory of vibrations, and in the studies of human speech.

The purpose of this introductory article is to give the reader the first quite general idea of commutative harmonic analysis and help him to orient himself, even vaguely, in this enormous subject while more elaborate treatment can be found in other articles of this and the following volumes.

Commutative harmonic analysis is situated at the intersection of a number of mathematical theories. One expert is inclined to treat Fourier series as a very special case of his beloved general theory of orthogonal systems. Another one, fascinated by spectral theory of differential operators, will say that Fourier analysis is just an investigation of the operator $\frac{d^2}{dx^2}$ under very special boundary conditions. The third will say that Fourier transform is only one out of many integral transforms used in analysis. The fourth feels that Fourier analysis is just an illustration of the theory of commutative Banach algebras .... And yet, all of them will be right since commutative harmonic analysis can be in many ways incorporated into various enveloping general theories. However, with each such inclusion it loses some of its important features, loses its original face. In this article, guided mainly by its classical appearance, we shall try to isolate and emphasize only those facts which are characteristic for commutative harmonic analysis as an independent discipline and shall demonstrate some of its connections with other parts of mathematics.

Together with differential and integral calculus and analytic geometry, elements of harmonic analysis now form essential part of each more or less thorough "Course of Higher Mathematics". The reader of this volume is most certainly familiar with the basic facts of Fourier Analysis. We, nevertheless, will have to recall those facts for at least to agree on notations. Also, we suggest (in the first chapter) a certain particular version of the "absolutely minimal" course on Fourier series and even present complete proofs. (Later on, the proofs are given less and less often, mostly in examples. Only some important constructions are presented in full detail). Our "minimal course" argues against generally accepted presentations which are still influenced by an old "string dispute" and where problems of pointwise convergence occupy the central place at the expense of the main topic: operational properties of the Fourier transform. We have dared to avoid the Dirichlet kernel in our "course" and constructed it entirely from the viewpoint of distribution theory. We understand, however, that inclusion of this theory into analysis courses at universities and engineering schools still poses a methodological problem. The second chapter is devoted to the Fourier transform in $\mathbb{R}^d$. Here, the presentation becomes more concise. The notions of convolution and a translation-invariant (t.i.) operator play the main role in Chapter 2. They are illustrated by examples of a fairly diverse origin. (cf. § 4). In the subsequent sections we give an outline of S'-theory and L^2-theory of Fourier transform in $\mathbb{R}^d$. In § 5 it appears as a device for an investigation of t.i. operators introduced earlier.

The statements (∗) and (∗∗) in the beginning are naive not only because of their imprecision but also because they do not yet reflect the fact that harmonic analysis (in its various forms) appears in presence of some "action" (more often, group of transformations) and its objects are not that much of individual functions but "trajectories" swept out by those functions under the above mentioned "action". The major merit of "harmonics" is simplicity of their reaction to the "action". This ideology is developed in the third chapter, which contains a brief outline of harmonic analysis on groups. A historical survey (Ch. 4) is followed by the final chapter. In the first four Chapters we have tried to emphasize those aspects of our subject which are especially important for other disciplines, and, therefore, have been dealing with very simple, "non-special" notions. The fifth chapter, together with a discussion of general and very important concepts of spectrum and spectral analysis-synthesis, also contains a brief list of much more special topics, which belong to the treasury of commutative harmonic analysis as a subject of its own. Those topics are only listed there. A detailed treatment, a discussion of their current status and statements of related contemporary problems can be found in other articles of this and the following volumes.

In conclusion I want to thank E.M. Dyn'kin, B. Jöricke A.A. Kirillov, S.V. Kislyakov, N.K. Nikol'skij, A.N. Podkorytov, M.V. Rudel'son and A.M. Vershik for numerous critical remarks which I have tried to incorporate in the text. I am grateful to N.K. Nikol'sksh for writing §§ 3.2–3.3 of Chapter 5, and to S.V. Khrushchev and B.A. Samokish for valuable consultations.

Chapter 1
A Short Course of Fourier Analysis of Periodic Functions

§ 1. Translation-Invariant Operators

1.1. The Set up. We will be talking about the space of distributions (generalized functions) $\mathscr{D}'$ on the unit circle $\mathbb{T} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C}: |\zeta| = 1\}$. (It should be denoted $\mathscr{D}'(\mathbb{T})$ but, for the sake of brevity, in this section we will be omitting the symbol $\mathbb{T}$ when speaking of spaces of functions defined on $\mathbb{T}$. So, e.g., the letter C will stand for the space $C(\mathbb{T})$ of all functions continuous on $\mathbb{T}$). Recall the definition of the space $\mathscr{D}'$.

Let f be a function defined on the circle $\mathbb{T}$. (The word "function" will always mean "complex-valued function"). If the function $x \to f(e^{ix})$ defined on the real axis $\mathbb{R}$ is differentiable at point x_0, $x_0 \in \mathbb{R}$, we shall say that f is differentiable at the point ζ_0, $\zeta_0 = e^{ix_0}$ and will write

$$(Df)(\zeta_0) = \frac{d}{dx}(f(e^{ix})|_{x=x_0}.$$

If $(Df)(\zeta_0)$ exists for all ζ_0, $\zeta_0 \in \mathbb{T}$, and $Df \in C$, then f is said to belong to the class C^1. Similarly, one can define classes C^p, $p = 2, 3, \ldots : f \in C^p \Leftrightarrow D^p f \in C$; $C^0 \stackrel{\text{def}}{=} C$. Set

$$\|f\|_C = \max\{|f(\zeta): \zeta \in \mathbb{T}\}(f \in C), \qquad \|f\|_{C^p} = \sum_{0 \leqslant j \leqslant p} \|D^j f\|_C (f \in C^p),$$

$\mathscr{D} = \bigcap_{p=0}^{\infty} C^p$. Let us agree to say that sequence $\{\alpha_j\}$ $(j \in \mathbb{N})$ of functions of class $\mathscr{D}$ tends to zero in $\mathscr{D}$ if $\lim_{j\to\infty} \|\alpha_j\|_{C^p} = 0$ $(p = 0, 1, \ldots)$.

Definition. A *distribution*, or a *generalized function* (on the circle $\mathbb{T}$) is a linear functional φ, defined on $\mathscr{D}$ and continuous in the sense[1] that $\lim_{j\to\infty} \varphi[\alpha_j] = 0$ for any sequence $\{\alpha_j\}$ $(j \in \mathbb{N})$ of functions in class $\mathscr{D}$ converging to zero in $\mathscr{D}$. The set of all distributions will be denoted by the symbol $\mathscr{D}'$.

It is wasy to show that for every distribution φ one can find numbers $p(\varphi) \in \mathbb{Z}_+$ and $K(\varphi) \in (0, +\infty)$ such that

$$|\varphi[\alpha]| \leqslant K(\varphi)\|\alpha\|_{C^{p(\varphi)}}$$

for any function α, $\alpha \in \mathscr{D}$.

An important class of distributions (although far from exhausting $\mathscr{D}'$) is formed by the set M of all (complex-valued) measures on $\mathbb{T}$. (We call measure a countably additive function defined on a system $\mathscr{B}$ of all Borel subsets of $\mathbb{T}$). Every measure μ, $\mu \in M$ induces a distribution φ_μ defined by the following formula.

$$\varphi_\mu[\alpha] = \int_{\mathbb{T}} \alpha \, d\mu \qquad (\alpha \in \mathscr{D}).$$

The mapping $\mu \to \varphi_\mu$ is one-to-one and hence, we shall not distinguish measure μ from the distribution φ_μ, and consider $M \subset \mathscr{D}'$. In particular, $\mathscr{D}'$ contains the delta-function δ, the unit mass concentrated at point $1 (\delta[\alpha] = \alpha(1), \alpha \in \mathscr{D})$.

Together with M, $\mathscr{D}'$ contains all spaces $L^p(= L^p(\mathbb{T}, m))$, $p \in [1 + \infty]$, where m is the normalized Lebesgue measure on the circle $\mathbb{T}$ (i.e., $m(E)$ is the length of a Lebesgue measurable subset E on $\mathbb{T}$ divided by 2π). In fact, every function f, $f \in L^1$, induces measure μ_f, $\mu_f \in M$ by

$$\mu_f(E) = \int_E f \, dm \qquad (E \in \mathscr{B})$$

and can be identified with it (i.e., from equality $\mu_f = \mu_g$ it follows that f and g coincide m-almost everywhere on $\mathbb{T}$).

---

[1] Here, $\varphi[\alpha]$ is the value taken by the functional φ, $\varphi \in \mathscr{D}'$ at an element α of the space $\mathscr{D}$. If φ is identified with a function of class L^1 (see below), then the symbol $\varphi(\zeta)$ means value of the function φ at point ζ on the circle.

Sequence of distributions $\{\varphi_j\}$ $(j \in \mathbb{N})$ is said to converge to the distribution φ, if $\lim_{j\to\infty} \varphi_j[\alpha] = \varphi[\alpha]$ for each function α, $\alpha \in \mathscr{D}$. Continuity of a linear operator L mapping space $\mathscr{D}'$ into itself will mean that $\lim_{j\to\infty} L(\varphi_j) = 0$ for any sequence of distributions $\{\varphi_j\}$ $(j \in \mathbb{N})$ converging to zero in $\mathscr{D}'$.

The set $\mathscr{D}$ is dense in $\mathscr{D}'$ in a sense that every distribution is the limit of a sequence of functions of class $\mathscr{D}$ (see § 1.3, below).

We shall also encounter distributions on the d-dimensional torus $\mathbb{T}^d \overset{\text{def}}{=} \underbrace{\mathbb{T} \times \cdots \times \mathbb{T}}_{d}$ $(\subset \mathbb{C}^d)$. In this case, $\mathscr{D}$ will denote the space of all functions α infinitely differentiable on the torus $\mathbb{T}^d$ (i.e., the function $(x_1, \ldots, x_d) \to \alpha(e^{ix_1}, \ldots, e^{ix_d})$ of d real variables is infinitely differentiable). $\mathscr{D}'$ is the space of all linear functionals defined and continuous on $\mathscr{D}$.

1.2. Object of Investigation. Translation-invariant (t.i.) linear operators play an important role in mathematical analysis and its applications. Avoiding precise definitions at the moment, let us just point out the main property of a t.i. operator L acting on a certain space of functions defined on $\mathbb{R}^d$:

$$L(f_h) = (L(f))_h \qquad (h \in \mathbb{R}^d),$$

where $f_h(x) \overset{\text{def}}{=} f(x + h)(x, h \in \mathbb{R}^d)$.

We will give below (see §§ 3.3, 3.4) numerous concrete examples of t.i. operator. At this point let us only note that all linear differential operators and finite difference operators with constant coefficients, and also many commonly used integral operators are translation-invariant.

The name of our subject ("commutative harmonic analysis") itself suggests a certain object of analysis although it does not name it explicitly. The adjective "harmonic"[2] points out not an object but the method of investigation. Roughly speaking the method consists of expansion of t.i. operators with respect to their eigenfunctions, i.e., "harmonics". Although basic, this method is far from being unique and more and more often nowadays any result, even obtained without using any "harmonics" but concerned with t.i. operators, is referred to harmonic analysis. For example, a broad theory of singular integral operators is considered to be a part of harmonic analysis, although some of its most significant achievements are obtained without using Fourier transform (i.e., without "harmonics") and even by getting rid of it altogether and turning to completely different methods. Referring to the contemporary studies Herz has wittily remarked that "... harmonic analysis has more to do with harmonic functions than trigonometric series" (Bull. Amer. Math. Soc., 1982, 7 No. 2, p. 422). So, t.i. operators is the main topic of harmonic analysis and its major aim. And this is where we start out, restricting ourselves at first to t.i. operators acting in spaces of 2π-periodic functions of a real variable or, which is the same, in spaces of functions defined

[2] The meaning of the adjective "commutative" (which is often omitted for the sake of brevity) will be discussed later, in Chapter 3.