Postgraduate Textbook Recommended by the Education Committee of Hunan Province

Introduction to Continuum Mechanics

Zhang Chunyuan Zhang Weimin

(连续介质力学引论)

Science Press
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Preface

This book is designed to form a one-semester course of sixty class hours in continuum mechanics at the first-year graduate level. The book is based upon notes of courses I have taught over the past twenty years to postgraduate students of specialties in Rheology, Solid Mechanics and Metal Materials at Xiangtan University. I wrote it in English in order to enhance the English level of the Chinese students. While students at first might find this a bit difficult, I believe that with the additional efforts required they will be requited with the resulting gain in ability.

The book presents an introduction to the theories of continuum mechanics. These theories are important, because not only they are applicable to a majority of the problems in continuum mechanics arising in practice, but also they form a solid base upon which one can readily studies theories of mechanics that are more complex. Further, although attention is limited to the classical theories, the treatment is modern with a major emphasis on foundations and structures. The reader will soon find that he needs to do some work on the side to fill in details that are omitted from the text. Furthermore, many of the important points are in the problems assigned for solution by the student. The reader who does not at least try to solve a good many of the problems is likely to miss most of the point.

Throughout this book, we will use the tensor notations that used by Guo Zhongheng[1], which is, in some aspects, different from those used by some foreign scholars as Truesdell[2,3], etc. It is presumed that readers are already familiar with the theory of tensors. Many of the result from theory of tensors will merely recall here and then applied. Concise notes on theory of tensors, which have the same notations of this book and originally appeared in my VISCOELASTIC FRACTURE MECHANICS[4,5], are present in Appendix A for reference.

The main part of the book has been written by Zhang Chunyuan with the help of Prof. Zhang Weimin and the Examples and the Exercises are due to Zhang Weimin.

This book was supported by the Foundation of Hunan Education Committee, the Foundations of Xiangtan University and Key Science of Mechanics in Xiangtan University. I own so much to so many of colleagues, friends, and students. I gratefully acknowledge Prof. Yang Tingqing, Prof. Zhou Yichun, Prof. Zhang Ping and many others for their comments and helps. I also wish to thank many of my students, without their discussions, the book would not have taken this form. Finally, I am extremely grateful to my wife, Prof. Ning Yaqin and my parents for their encouragement, discussions and helps.

Zhang Chunyuan

Xiangtan, Hunan April 14, 2003

CONTENTS

1 INTRODUCTION

1.1 Rational Continuum Mechanics

The sixth of problems that famous mathematician Hilbert set for the twentieth century to solve was the formulation of an *axiomatic structure* for physics, and especially for mechanics. Since the efforts of Noll, Coleman and many other authors in the past fifty years, a beautiful flower of rational mechanics finally comes into bloom. *Rational continuum mechanics* is a new course of mechanics of a complete *axiom system* that set-up based upon the production and a wealth of experiential facts. Following the basic laws that must be obeyed by the motion of materials and using the strict logical thinking and inference, it studies the general laws of the *motion* and *deformation* of materials. It avoids the microstructure of the matters, and set-up a theory of continuous field. It reflects the characteristic of the modern mechanics, which extends the study from method of analysis to that of synthesis, from problems of *Equilibrium State* to that of Non-equilibrium State, from linear problems to nonlinear problems and from macroscopic continuous bodies to bodies having internal structures. It is a cognition leap from the approximate theory to rigorous theory and that from accumulation of knowledge to mathematical abstraction. It strongly affects the traditional education and research works and greatly enhances the capacity of solving practical problem by mechanics. This course began in 1945 by Reiner and Rivlin. Afterwards, it has great development due to the efforts of Truesdell, Noll, Coleman and others. In China, Qian Weichang held the first national symposium of Rational Mechanics in Langzhou in 1979[6].

1.2 Rational Continuum Mechanics and Rheology

Rheology and Rational Mechanics are twin courses developed simultaneously. Rheology focuses its attention at the research of the nonlinear viscoelastic responses of matters. It mainly discusses the *constitutive relations* of materials. While Rational Mechanics take the formation of mathematics system of continuum mechanics for its aim. Both of them research on the common properties of materials, which do not distinguish solids and liquids.

1.3 System of Rational Mechanics

Mechanics does not study natural things directly[2]. Instead, it considers *bodies*, which are mathematical concepts designed to abstract some common features of many nature things. One such feature is the *mass* assigned to each body. Bodies are always found to occupy some *place*. The theory of places, which is called *geometry*, was created long ago thus lies ready to hand for application in mechanics. The change of place undergone by a body in the course of time is called the *motion* of that body, and description of motion, or *kinematics*, is the second part of the foundation of mechanics. Third, motions of bodies are conceived as resulting from or at least being invariably accompanied by the action of *forces*. Fourth, the gain and loss of heat give rise to the concepts of the *energy*, *temperature*, and *calorie* of a body. Thus, mechanics is a *mathematical model*, or better, an infinite class of models, for certain aspects of nature.

Kinematics, then, being presumed, mechanics rests upon four substructures: theories of bodies, forces, energies, and calories, in connection with places, times, and temperatures. These substructures provide the concepts mechanics is to connect. Relations among them are of two kinds: the general ones, common to all the bodies entering a given branch of mechanics, and particular ones, which distinguish one class of such bodies from another. The former kind constitutes two theories: *statics*, which compares putative equilibrium, and *dynamics*, which refers to motions of all sorts. Relations of the later kind, which define particular bodies, are called *constitutive*.

1.3.1 Primitive elements

The *primitive elements* of mechanics are "bodies", "motions" and "forces"[3]. These elements are governed by assumptions or laws, which describe mechanics as a whole.

1.3.2 Basic laws

The primitive elements must obey the *basic laws* commonly. The laws abstract the common feature of all mechanical phenomena. The basic laws include: *conservation of mass*, *balance of momentum*, *balance of moment of momentum*, *conservation of energy* and balance of entropy or equivalently, *Clausius-Duhem inequality* (*the second law of thermodynamics*).

In cases of continuous fields, they become the *field equations* and in cases of non-continuous fields, they become the *discontinuity conditions*.

1.3.3 Constitutive relations

Based on the experimental evidence and guided by the *basic principles* the *constitutive equations* describing materials having different characteristic can be found. Constitutive equations serve as models for different kinds of bodies. They define ideal materials, intended to represent aspects of the different behaviors of various materials in a physical world subjected to simple and overriding laws. The basic principles include: *principle of determinism*, *principle of local action* and *principle of frame-indifference*. The improvement of model characterizes that the cognition of human being is deepened.

Field equations (or discontinuity conditions) and constitutive equations constitute the *governing equations*, which can be solved under the *initial conditions* and *boundary conditions*.

Primitive elements, basic laws and the constitutive equations constitute the *system of rational continuum mechanics*.

In order to establish the mathematical formulation of any physical phenomenon that take place in a material body[7] we need to establish a correspondence between the elements of the physical body and the *idealized mathematical body*. The mathematical axioms and operations can then be used to study various problems, which can be translated back to the physical body with rules of correspondence established. The predictions so made can be compared with observations testing the limitations of the theory. All mechanical phenomena are considered to be the result of *body points* with mass under a variety of external conditions. The mathematical idealization to bodies can be made in the following broadest sense: The body points $\{P\}$ of a body constitute the elements of a set, called the *material body* $\mathscr{B}$. These elements are considered to be known a priori from certain physical considerations that are fundamental to the structure of the mathematical theory of the physical phenomena, which we wish to study. The set $\mathscr{B}$ is considered to be a subset of the *universal set* $\mathscr{U}$. This is the frame of reference or the *universe* for the discussion of $\mathscr{B}$. The *complement* of $\mathscr{B}$, denoted by $\mathscr{B}'$, is the set of elements, which are not in $\mathscr{B}$ (Fig. 1.3-1). This may be envisioned as the space surrounding the body, which may contain other bodies as well. Both $\mathscr{B}$ and $\mathscr{B}'$ may contain subsets. Further, we shall introduce some coherence (geometric structure) to the elements of these sets so that these sets can be organized to a space. For mathematical operations, it is also necessary to establish *rules of operations*. This corresponds to the physical laws. This program then establishes the ideal mathematical body corresponding to the physical body. Summarizing we have:

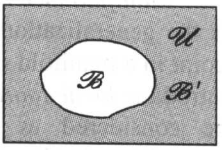

Fig.1.3-1 Body

Physical body	$\leftrightarrow$ set $\mathscr{B}$
Body points	$\leftrightarrow$ elements $\{P\}$ of set $\mathscr{B}$
Frame of reference	$\leftrightarrow$ the universal set $\mathscr{U}$
Structure of body	$\leftrightarrow$ *topological structure* of $\mathscr{B}$
Physical laws	$\leftrightarrow$ *operation rules* among $\{P\}$ and among $\mathscr{B}$ and $\mathscr{U}$ or $\mathscr{B}'$

A physical theory is concerned with the evolution of the topological structure of the body and the interrelations between the body and the external world $\mathscr{B}'$. The difference in the nature of the physical phenomena arises from the difference in the nature of the structure of the set $\mathscr{B}$ and the operations to which the elements of $\mathscr{B}$ are subject.

2 KINEMATICS

Chapter 2 deals with the deformation and kinematics of continua. This is a pure geometrical problem and has no concern with the course of deformation and motion and the property of the continua. In the first nine sections of this chapter, we present an analysis of the deformation of material bodies and develop relationships between the parts of the deformed and undeformed body when the displacement of the body is prescribed point-wise. The remaining part of this chapter discusses the changes of various variables of kinematics with time.

2.1 *Bodies, Configurations and Motions*

2.1.1 Bodies

Manifold (see A.9) is an important concept in modern mathematics and physics. It is the generalization of Euclidean space. Roughly speak, in the neighborhood of any point in a manifold is *homeomorphic* to an open set of Euclidean space, so that we may introduce *local coordinate systems*. A manifold (For example, a *spherical surface*) can be considered as a topological space glued by many pieces of "*Euclidean spaces*"(small planes). It is just a *higher-dimensional* analogue of a smooth curve or surface.

A body $\mathscr{B}$ is a manifold of body points, denoted by P. Body is a set of body points. In continuum mechanics the *body manifold* is assumed to be smooth, that is, a *diffeomorphism* of a domain in Euclidean space. Thus, by assumption, the body point P can be set into one-to-one correspondence with triples of real numbers X^1, X^2, X^3, where the X^k run over a finite set of closed intervals. Such triples are sometimes called "*intrinsic coordinates*" of the body point, but we shall not need to use them explicitly. The mapping from the manifold to the domain is assumed *differentiable* as many time as desired, usually two or three times, without further mention.

The body $\mathscr{B}$ is assumed also to be a *σ-finite measure space* with *non-negative measure* $M(\mathscr{P})$ defined over a σ-ring of subsets $\mathscr{P}$, which are called the parts of $\mathscr{B}$. Henceforth any subset of $\mathscr{B}$ to which we shall refer will be assumed to be measurable. The measure $M(\mathscr{P})$ is called the *mass distribution* in $\mathscr{B}$.

The two fundamental properties of a continuum body $\mathscr{B}$ are:

1. $\mathscr{B}$ consists of a finite number of parts that can be mapped smoothly onto cubes in Euclidean space.

2. $\mathscr{B}$ is a measure space.

Bodies are available to us only in their *configurations*, the regions they happen to occupy in Euclidean space at some time. These configurations are not to be confused with the bodies themselves.

When we research the *non-compatible* physical problems, such as fracture, welding, wound and healing, the concept of manifold is more important.

2.1.2 Configurations

A body $\mathscr{B}$ (the dotted line in Fig. 2.1-1) can be mapped smoothly onto a domain in Euclidean space. The region it happens to occupy in Euclidean space at some time is called the configuration (the solid lines) of the body. Often it is convenient to select one particular configuration and refer every thing concerning the body to that configuration, which need be only a possible one, not one ever occupied by the body. Such a configuration is called the *reference configuration*. Generally, we will use the region occupied by the body at t = 0 in Euclidean space as reference configuration. This configuration is also called the *initial configuration*. Let $\boldsymbol{\kappa}$ be such a configuration. Then the *mapping*

$$\boldsymbol{P}=\boldsymbol{\kappa}(P), \qquad P\in\mathscr{B} \tag{2.1-1a}$$

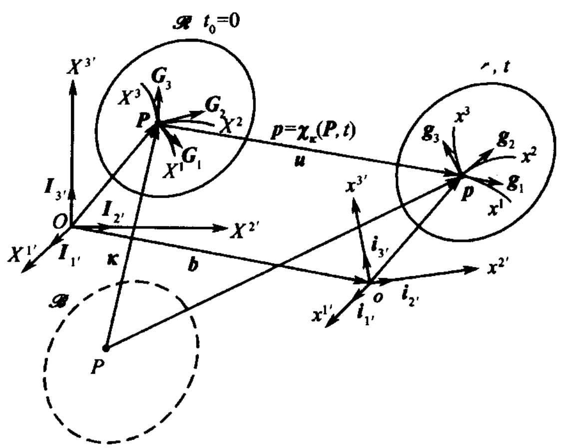

Fig. 2.1-1 Body, configuration and motion

gives the place vector $\boldsymbol{P}$ referred to the origin O and occupied by the body point P in the configuration $\boldsymbol{\kappa}$. Since this mapping is smooth, by assumption,

$$P=\boldsymbol{\kappa}^{-1}(\boldsymbol{P}), \qquad \boldsymbol{P}\in\mathscr{R}, \qquad \mathscr{R}=\boldsymbol{\kappa}(\mathscr{B}). \tag{2.1-1b}$$

2.1.3 Motions and coordinates

In order to describe the deformation and motion of the body, we will use two independent *curvilinear coordinate systems*: $\{X^K\}$ in $\mathscr{R}$ and $\{x^k\}$ in $\boldsymbol{\checkmark}$ referred to two independent Cartesian coordinate systems $\{X^{K'}\}$ and $\{x^{k'}\}$, respectively. A particular body point P at $t=0$ in $\{X^{K'}\}$ has *radius vector* $\boldsymbol{P}$ in $\{X^{K'}\}$ and has coordinates X^K in

$\{X^K\}$. The coordinates X^K can be viewed as the symbol of the body point P. X^K remains its value in the whole motion of the body. So that we may call X^K (K = 1, 2, 3) the *material coordinates*† and $\{X^K\}$ the *material coordinate system.*

Since a body can be mapped smoothly onto a domain, it can be mapped onto any other topological equivalent of that domain. A sequence (with the time t as a parameter) of such mappings

$$\mathbf{p}=\boldsymbol{\chi}(P,t),\quad P\in\mathscr{B},\quad t\in\mathscr{R}^{+},\quad \mathscr{r}=\boldsymbol{\chi}(\mathscr{B},t) \qquad (2.1\text{-}2)$$

is called a *motion* $\boldsymbol{\chi}$, where $\mathscr{R}^+$ is the positive real axis. The configuration at time t is called the *current configuration* $\mathscr{r}$.

Substituting (2.1-1) into (2.1-2), we have

$$\mathbf{p}=\boldsymbol{\chi}(\boldsymbol{\kappa}^{-1}(\mathbf{P}),t)=\boldsymbol{\chi}_{\kappa}(\mathbf{P},t),\quad \mathscr{r}=\boldsymbol{\chi}_{\kappa}(\mathscr{R},t).\ \mathbf{P}=\boldsymbol{\chi}_{\kappa}^{-1}(\mathbf{p},t),\quad \mathscr{R}=\boldsymbol{\chi}_{\kappa}^{-1}(\mathscr{r},t). \qquad (2.1\text{-}3a)$$

or simply

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{P},\ t), \qquad \boldsymbol{P} = \boldsymbol{P}(\boldsymbol{p},\ t), \qquad (2.1\text{-}3b)$$

or in coordinate notation

$$x^k = x^k(X^K,\ t), \qquad X^K = X^K(x^k,\ t). \qquad (2.1\text{-}3c)$$

x^k (k = 1, 2, 3) are called the *spatial coordinates* and $\{x^k\}$ the *spatial coordinate system.*

Motion can also be said to be the sequence of mappings from the initial configuration to the current configuration with the parameter t. At t = 0, the body $\mathscr{B}$ occupies the region $\mathscr{R}$ with surface $\mathscr{S}$ and volume $\mathscr{V}$ (see Fig. 2.1-1) referred to a fixed Cartesian coordinate system $\{X^{K'}\}$. At time t the body moves into the region $\mathscr{r}$ with surface $\mathscr{s}$ and volume $\mathscr{v}$ referred to the fixed Cartesian coordinate system $\{x^{k'}\}$. Each point $\boldsymbol{P}$ (X^k) in $\mathscr{R}$ at t = 0 moves onto the point $\boldsymbol{p}$ (x^k) in $\mathscr{r}$ at time t.

Axiom of Continuity

In continuum mechanics, the mapping $\boldsymbol{\chi}_{\kappa}$ is assumed to be *one-to-one correspondence*, $\boldsymbol{\chi}_{\kappa}$ and $\boldsymbol{\chi}_{\kappa}^{-1}$ are *continuous and differentiable.* Such a mapping is called a *homeomorphism.* The *Jacobian*

$$J\equiv\det(x^k{}_{,K})=\begin{vmatrix}\partial x^1/\partial X^1 & \partial x^1/\partial X^2 & \partial x^1/\partial X^3\\ \partial x^2/\partial X^1 & \partial x^2/\partial X^2 & \partial x^2/\partial X^3\\ \partial x^3/\partial X^1 & \partial x^3/\partial X^2 & \partial x^3/\partial X^3\end{vmatrix}>0 \qquad (2.1\text{-}4)$$

at all points of $\mathscr{B}$.

It expresses that the matter is *indestructible*, that is, no region of positive, finite volume of matter is deformed into a zero or infinite volume. Alternatively, the matter is *impenetrable*, that is, the motion can only carry every region into a region, every surface into a surface and every curve into a curve. No one portion of matter can penetrate into another.

The motion that satisfies the axiom of continuity is called an *admissible motion.* However, for the problems of fracture, welding, wound and healing, the mapping will no longer be homeomorphic. The image of one point may be two points or the images

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† It should be noted that in curvilinear coordinate systems, X^K or x^k are not components of a vector. For the position vectors of $\boldsymbol{P}$ and $\boldsymbol{p}$, we have $\boldsymbol{P}\neq X^K\boldsymbol{G}_K$, $\boldsymbol{p}\neq x^k\boldsymbol{g}_k$ (see (2.1-35)) in spite of $\mathrm{d}\boldsymbol{P}=\mathrm{d}X^K\boldsymbol{G}_K$, $\mathrm{d}\boldsymbol{p}=\mathrm{d}x^k\boldsymbol{g}_k$ (see (2.1-9)). For example, in cylindrical coordinates $\boldsymbol{p}\neq r\boldsymbol{g}_r+\theta\boldsymbol{g}_\theta+z\boldsymbol{g}_z$, but $\mathrm{d}\boldsymbol{p}=\mathrm{d}r\boldsymbol{g}_r+\mathrm{d}\theta\boldsymbol{g}_\theta+\mathrm{d}z\boldsymbol{g}_z$

of two points may be one point.

Henceforth, the coordinates x^k for the deformed body are called *spatial* or *Eulerian* whereas the coordinates X^K of the nature state are called *material* or *Lagrangean.* The quantities referred to the spatial coordinates will be denoted by small Latin kernel letters and their components by small Latin *indices.* The quantities referred to the material coordinates will be denoted by large Latin kernel letters and their components by large Latin indices.

2.1.4 Base vectors, metric tensors, shifters, displacements

Base vectors, metric tensors

The place vector $\boldsymbol{P}$ of point P in $\mathscr{R}$ and the corresponding place vector $\boldsymbol{p}$ of point P in $\boldsymbol{\mathscr{r}}$, respectively, are

$$\boldsymbol{P} = X^{K'} \boldsymbol{I}_{K'}, \qquad \boldsymbol{p} = x^{k'} \boldsymbol{i}_{k'}. \tag{2.1-5}$$

$$\boldsymbol{I}_{K'} \cdot \boldsymbol{I}_{L'} = \delta_{K'L'}, \qquad \boldsymbol{i}_{k'} \cdot \boldsymbol{i}_{l'} = \delta_{k'l'}. \tag{2.1-6}$$

The non-coplanar base vectors $\boldsymbol{G}_K$ and $\boldsymbol{g}_k$ can be defined, respectively, as

$$\boldsymbol{G}_K(\boldsymbol{P}) = \frac{\partial \boldsymbol{P}}{\partial X^K} = \frac{\partial \boldsymbol{P}}{\partial X^{K'}}\frac{\partial X^{K'}}{\partial X^K} = \frac{\partial X^{K'}}{\partial X^K}\boldsymbol{I}_{K'} = X^{K'}{}_{,K}\boldsymbol{I}_{K'}, \tag{2.1-7}$$

$$\boldsymbol{g}_k(\boldsymbol{p}) = \frac{\partial \boldsymbol{p}}{\partial x^k} = \frac{\partial \boldsymbol{p}}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^k} = \frac{\partial x^{k'}}{\partial x^k}\boldsymbol{i}_{k'} = x^{k'}{}_{,k}\boldsymbol{i}_{k'}. \tag{2.1-8}$$

The differential vector elements $\mathrm{d}\boldsymbol{P}$ at $\boldsymbol{P}$ and $\mathrm{d}\boldsymbol{p}$ at $\boldsymbol{p}$, respectively, are

$$\mathrm{d}\boldsymbol{P} = \frac{\partial \boldsymbol{P}}{\partial X^K}\mathrm{d}X^K = \boldsymbol{G}_K \mathrm{d}X^K, \quad \mathrm{d}\boldsymbol{p} = \frac{\partial \boldsymbol{p}}{\partial x^k}\mathrm{d}x^k = \boldsymbol{g}_k \mathrm{d}x^k. \tag{2.1-9}$$

The reciprocal contravariant base vectors $\boldsymbol{G}^K$ and $\boldsymbol{g}^k$ are defined as

$$\boldsymbol{G}^K \cdot \boldsymbol{G}_L = \delta^K{}_L, \qquad \boldsymbol{g}^k \cdot \boldsymbol{g}_l = \delta^k{}_l. \tag{2.1-10}$$

The covariant metric tensors are defined as

$$G_{KL}(\boldsymbol{P}) = \boldsymbol{G}_K \cdot \boldsymbol{G}_L = X^{K'}{}_{,K} X^{L'}{}_{,L}\, \delta_{K'L'} = X^{K'}{}_{,K} X^{K'}{}_{,L},$$

$$g_{kl}(\boldsymbol{p}) = \boldsymbol{g}_k \cdot \boldsymbol{g}_l = x^{k'}{}_{,k} x^{l'}{}_{,l}\, \delta_{k'l'} = x^{k'}{}_{,k} x^{k'}{}_{,l}. \tag{2.1-11}$$

The contravariant $\boldsymbol{\mathscr{r}}$ components G^{KL} and g^{kl} of the metric tensors satisfy the following equations

$$G^{KM} G_{ML} = \delta^K{}_L, \qquad g^{km} g_{ml} = \delta^k{}_l. \tag{2.1-12}$$

The solutions of these equations are

$$G^{KL} = (\text{cofactor of } G_{KL})/\det G_{KL} = (1/2G)\, e^{KPQ} e^{LRS} G_{PR} G_{QS}, \tag{2.1-13}$$

$$g^{kl} = (\text{cofactor of } g_{kl})/\det g_{kl} = (1/2g)\, e^{kpq} e^{lrs} g_{pr} g_{qs}, \tag{2.1-14}$$

where

$$G = \det G_{KL} = (1/3!)\, e^{KPQ} e^{LRS} G_{KL} G_{PR} G_{QS}, \tag{2.1-15}$$

$$g = \det g_{kl} = (1/3!)\, e^{kpq} e^{lrs} g_{kl} g_{pr} g_{qs}, \tag{2.1-16}$$

The reciprocal base vectors $\boldsymbol{G}^K$ and $\boldsymbol{g}^k$ are calculated by

$$\boldsymbol{G}^K = G^{KL} \boldsymbol{G}_L, \qquad \boldsymbol{g}^k = g^{kl} \boldsymbol{g}_l. \tag{2.1-17}$$

On the other hand, we have

$$\boldsymbol{G}^K = X^K{}_{,K'} \boldsymbol{I}_{K'}, \qquad \boldsymbol{g}^k = x^k{}_{,k'} \boldsymbol{i}_{k'}. \tag{2.1-18}$$

Thus, we have

$$G^{KL} = X^K{}_{,K'} X^L{}_{,K'}, \qquad g^{kl} = x^k{}_{,k'} x^l{}_{,k'} \tag{2.1-19}$$

and

$$G = |G_{KL}| = |X^{K'}{}_{,K}|^2, \qquad g = |g_{kl}| = |x^{k'}{}_{,k}|^2, \tag{2.1-20}$$
$$G^{-1} = |G^{KL}| = |X^{K}{}_{,K'}|^2, \qquad g^{-1} = |g^{kl}| = |x^{k}{}_{,k'}|^2. \tag{2.1-21}$$

Shifters

Shifters are two point tensors. We first give the definition of *two-point tensors.* Quantities that transform like tensors with respect to the indices K and k upon transformation of each of the two sets of coordinates, $\{x^k\}\rightarrow\{x^{k'}\}$ and $\{X^K\}\rightarrow\{X^{K'}\}$, are called two-point tensors. If

$$x^{k'} = x^{k'}(x^k), \qquad X^{K'} = X^{K'}(X^K) \tag{2.1-22}$$

are *differentiable coordinate transformations*, and if

$$T^{k'}{}_{K'}(p', P') = \frac{\partial x^{k'}}{\partial x^k}\frac{\partial X^K}{\partial X^{K'}}T^k{}_K(p, P) = A^{k'}_k A^K_{K'} T^k{}_K(p, P), \tag{2.1-23}$$

then $T^k{}_K(p, P)$ are the components of an absolute two-point tensor field $\overset{)(}{\boldsymbol{T}}(p, P)$.

$$\overset{)(}{\boldsymbol{T}}(p, P) = T^k{}_K(p, P)\boldsymbol{g}_k(p)\boldsymbol{G}^K(P). \tag{2.1-24}$$

Here we use notations "⟨" and "⟩" above the tensor $\boldsymbol{T}$ to denote the tensor in $\mathscr{R}$ and in $\mathscr{r}$, respectively. Deformation gradients and shifters are examples of two-point tensors. The *algebraic operation* of two-point tensors is the same as those of one-point tensors. The difference is that shifters appear in the dot product of two different sets of base vectors.

Now we may express the sets ($\boldsymbol{G}_K$, $\boldsymbol{G}^K$) and ($\boldsymbol{g}_k$, $\boldsymbol{g}^k$) in terms of one anther in a unique way

$$\boldsymbol{G}_K = g_K{}^k\boldsymbol{g}_k, \qquad \boldsymbol{G}^K = g^K{}_k\boldsymbol{g}^k, \qquad \boldsymbol{g}_k = g_k{}^K\boldsymbol{G}_K, \qquad \boldsymbol{g}^k = g^k{}_K\boldsymbol{G}^K, \tag{2.1-25}$$

where the coefficients

$$g^k{}_K = g_K{}^k = \boldsymbol{G}_K(P)\cdot\boldsymbol{g}^k(p) = g^{kl}G_{KL}g^L{}_l, \qquad g^K{}_k = g_k{}^K = \boldsymbol{G}^K(P)\cdot\boldsymbol{g}_k(p) = g_{kl}G^{KL}g_L{}^l,$$
$$g^{kK} = g^{Kk} = \boldsymbol{G}^K(P)\cdot\boldsymbol{g}^k(p) = G^{KL}g^{kl}g_{Ll}, \qquad g_{kK} = g_{Kk} = \boldsymbol{G}_K(P)\cdot\boldsymbol{g}_k(p) = G_{KL}g_{kl}g^{Ll} \tag{2.1-26}$$

are called *shifters*. The last equations in (2.1-26) imply that the indices can be raised and lowered by metric tensors. The *dyadic forms* of shifters are

$$\overset{)(}{\boldsymbol{I}} = g_{kK}\boldsymbol{g}^k\boldsymbol{G}^K = g^k{}_K\boldsymbol{g}_k\boldsymbol{G}^K = g_k{}^K\boldsymbol{g}^k\boldsymbol{G}_K = g^{kK}\boldsymbol{g}_k\boldsymbol{G}_K,$$
$$\overset{()}{\boldsymbol{I}} = g_{Kk}\boldsymbol{G}^K\boldsymbol{g}^k = g_K{}^k\boldsymbol{G}^K\boldsymbol{g}_k = g^K{}_k\boldsymbol{G}_K\boldsymbol{g}^k = g^{Kk}\boldsymbol{G}_K\boldsymbol{g}_k. \tag{2.1-27}$$

Since

$$\boldsymbol{G}_L\cdot\boldsymbol{G}^K = (g_L{}^l\boldsymbol{g}_l)\cdot(g^K{}_k\boldsymbol{g}^k) = g_L{}^kg^K{}_k,$$

we have

$$g_L{}^kg^K{}_k = g_L{}^kg_k{}^K = \delta_L{}^K, \qquad \text{or} \quad \overset{()}{\boldsymbol{I}}\cdot\overset{)(}{\boldsymbol{I}} = \overset{((}{\boldsymbol{I}}. \tag{2.1-28}$$

Since

$$\boldsymbol{g}_l\cdot\boldsymbol{g}^k = (g_l{}^L\boldsymbol{G}_L)\cdot(g^k{}_K\boldsymbol{G}^K) = g_l{}^Kg^k{}_K,$$

we have

$$g_l{}^Kg^k{}_K = g_l{}^Kg_K{}^k = \delta_l{}^k, \qquad \text{or} \quad \overset{)(}{\boldsymbol{I}}\cdot\overset{()}{\boldsymbol{I}} = \overset{))}{\boldsymbol{I}}. \tag{2.1-29}$$

We also have

$$\sqrt{G}=[\boldsymbol{G}_1\boldsymbol{G}_2\boldsymbol{G}_3]=[\overset{\rangle\langle}{\boldsymbol{I}}\cdot\boldsymbol{G}_1\ \overset{\rangle\langle}{\boldsymbol{I}}\cdot\boldsymbol{G}_2\ \overset{\rangle\langle}{\boldsymbol{I}}\cdot\boldsymbol{G}_3]=g_1^i g_2^j g_3^k[\boldsymbol{g}_i\boldsymbol{g}_j\boldsymbol{g}_k]=g_1^i g_2^j g_3^k e_{ijk}[\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3]$$

$$=\left|g_K^k\right|[\boldsymbol{g}_1\boldsymbol{g}_2\boldsymbol{g}_3]=\left|g_K^k\right|\sqrt{g}, \tag{2.1-30}$$

$$\sqrt{g}=\left|g_k^K\right|\sqrt{G}.$$

By means of these tensors, we can express components of a vector in two coordinate systems in terms of each other. For example, in two coordinate systems, $\{X^K\}$, $\{x^k\}$, a vector $\boldsymbol{v}$ can be written as

$$\boldsymbol{v}=V^L\boldsymbol{G}_L=v^k\boldsymbol{g}_k, \qquad (\text{or} \qquad \boldsymbol{v}=V_L\boldsymbol{G}^L=v_k\boldsymbol{g}^k). \tag{2.1-31}$$

By taking the dot product of $\boldsymbol{G}^K$ or $\boldsymbol{g}^k$ (or $\boldsymbol{G}_K$ or $\boldsymbol{g}_k$) and the above equation, we obtain the *translation rule of a vector*

$$V^K=g^K{}_kv^k, \qquad v^k=g^k{}_KV^K, \qquad V_K=g_K{}^kv_k, \qquad v_k=g_k{}^KV_K. \tag{2.1-32}$$

or

$$\overset{\langle}{\boldsymbol{v}}=\overset{\langle\rangle}{\boldsymbol{I}}\cdot\overset{\rangle}{\boldsymbol{v}}, \qquad \overset{\rangle}{\boldsymbol{v}}=\overset{\rangle\langle}{\boldsymbol{I}}\cdot\overset{\langle}{\boldsymbol{v}}. \tag{2.1-33}$$

The translation rule of vectors may be viewed as a special *transformation law of tensors*. The shifters g_K^k, g_k^K (we may drop the distinction between $g^k{}_K$ and $g_K{}^k$ or $g^K{}_k$ and $g_k{}^K$) may be viewed as special coordinate transformation factors. If $\overset{\langle\rangle}{\boldsymbol{I}}$ applies to a vector $\overset{\rangle}{\boldsymbol{v}}$ at point $\boldsymbol{p}$ in $\mathscr{r}$, then $\overset{\rangle}{\boldsymbol{v}}$ is shifted from $\boldsymbol{p}$ to the vector $\overset{\langle}{\boldsymbol{v}}$ at point $\boldsymbol{P}$ in $\mathscr{R}$. If $\overset{\rangle\langle}{\boldsymbol{I}}$ applies to a vector $\overset{\langle}{\boldsymbol{v}}$ at the point $\boldsymbol{P}$ in $\mathscr{R}$, then $\overset{\langle}{\boldsymbol{v}}$ is shifted from $\boldsymbol{P}$ to the vector $\overset{\rangle}{\boldsymbol{v}}$ at point $\boldsymbol{p}$ in $\mathscr{r}$. $\overset{\langle}{\boldsymbol{v}}$ or $\overset{\rangle}{\boldsymbol{v}}$ is the same vector. The only difference is that they are considered at different point. We may shift the two-order tensors in a similar way.

Displacements. From Fig. 2.3-1 we see that

$$\boldsymbol{u}=\boldsymbol{p}-\boldsymbol{P}+\boldsymbol{b} \tag{2.1-34}$$

where $\boldsymbol{u}$ is the *displacement vector*. $\boldsymbol{u}$ can be viewed as a vector in $\{X^K\}$ or $\{x^k\}$.

$$\boldsymbol{P}=X^{K'}\boldsymbol{I}_{K'}=P^K\boldsymbol{G}_K=P^Kg_K{}^k\boldsymbol{g}_k\neq X^K\boldsymbol{G}_K,$$

$$\boldsymbol{p}=x^{k'}\boldsymbol{i}_{k'}=p^k\boldsymbol{g}_k=p^kg_k{}^K\boldsymbol{G}_K\neq x^k\boldsymbol{g}_k, \tag{2.1-35}$$

$$\boldsymbol{u}=U^K\boldsymbol{G}_K=u^k\boldsymbol{g}_k.$$

2.2 Description of Motion

In continuum mechanics, we may use the *material* or *Lagrangean description* and the *spatial* or *Eulerian description*. The relative description will discuss in Section 2.9.2. Because of our hypotheses of smoothness, all are equivalent.

2.2.1 The material description or Lagrangean description

The *material description* or Lagrangean description takes $\boldsymbol{P}$ or X^K and t as independent variables. It is the description commonly used in modern works on continuum mechanics. In this description, the motion can be described by