Studies in Mathematics

Volume 1
STUDIES IN MODERN ANALYSIS

by R. C. Buck

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R. C. Buck, editor
University of Wisconsin

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PREFACE

With the appearance of this volume, the Mathematical Association of America has embarked upon a new publishing venture. The MAA Studies in Mathematics will bring to the members of the Association, and to the general mathematical community, expository articles at the collegiate and graduate level on recent developments in mathematics and the teaching of mathematics. We hope that these will help to overcome the communication barrier which has arisen as a natural consequence of the tremendous acceleration in mathematical development that has taken place, especially within the last twenty-five years.

We hope that these volumes of short papers will be used as a basis for seminars, reports, informal talks, and as supplementary material to provide background knowledge for both students and faculty. The range of topics will cover primarily the upper class and beginning graduate years; the volumes that are planned at present will have articles at different levels of difficulty, so that the spectrum of each volume is wide.

The need for expository articles of this nature has been long recognized. Indeed, the MAA set a precedent with its highly successful "What Is ——?" series that first appeared in the *Monthly* more than twenty years ago. Three years ago, Professor

Richard V. Andree of the University of Oklahoma conceived of the plan to reprint these articles in a single volume; it at once became clear that many would have to be rewritten to bring them up to date, while at the same time there were many fields of mathematics that were inadequately represented in the original collection. Taking this as a personal challenge, Andree approached a large segment of the mathematical community; succeeding where others failed, he managed to overcome the natural lethargy of research mathematicians, and amassed an outstanding collection of expository articles. These will form the core of the first few volumes of the Studies. We hope that the momentum which Andree has given this effort will not die, and that the present atmosphere, favorable to expository writing, will persist.

R. P. Dilworth
Chairman, Committee on Publications,
Mathematical Association of America

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INTRODUCTION

R. C. Buck

The four papers that are presented in this volume do not pretend to cover all of modern analysis. They are, however, representative; each discusses topics that are fundamental, both for pure and for applied analysis, and which indeed should be part of the experience of every practicing mathematician, regardless of his field. These papers also achieve something which seems more important to me, for they succeed in the more difficult task of conveying some of the *attitudes* that are characteristic of modern mathematicians.

It would be difficult to devise a concise definition of analysis that is broad enough to cover all that now carries this label. It would seem no longer appropriate to confine analysis to "the theory of functions" since algebra and topology have equal rights to claim this as their domain; indeed, algebra has been called the study of homomorphisms, and topology the study of continuous mappings. The dissolution of traditional boundaries between branches of mathematics is probably the most striking aspect of the modern period of development. In the field of analysis, this has shown itself in a growing concern for matters of structure, and the emergence of what can only be called *algebraic analysis* and *topological analysis*.

These trends are clearly visible in the selections that comprise this volume. In the papers by Stone and Goffman, the algebraic component is apparent. In looking at a continuous function, one remains conscious of the fact that it is at the same time a member of a class of functions forming a mathematical entity such as a linear space or an algebra. Alongside this, we cannot ignore the realization that topological ideas lie at the heart of the fundamental processes of analysis; this is indeed the theme of the paper by McShane, but it is implicit in that of Lorch as well. A modern research paper on partial differential equations may refer to "the compact-open topology," or to the equivalence of "the strong and weak topology"; a paper on function theory may speak of compact sets rather than normal families, and may indeed discuss the nature of closed ideals in an algebra of holomorphic functions.

This intermarriage of traditional analysis with its neighbors has not come about as a rational decision of its practitioners. At first sight, the change seemed to have been largely a matter of semantics; one adopted the terminology of algebra and topology solely as a convenience to describe briefly certain situations which arose frequently. But it soon became evident that the adoption of another viewpoint, another observation platform, gave a clearer vision; the introduction of techniques borrowed from other fields enabled the analyst to achieve both striking economies in proof, and vivid insights into classical phenomena. Two prime examples are included in these papers: the first is the treatment of the Weierstrass approximation theorem, as given in the paper by M. H. Stone; the second, included in the paper by Goffman, is the unified treatment of certain existence theorems that is made possible by the study of contraction mappings on complete metric spaces. An even more persuasive example, not included here, would be the chain of observations that start from the classical theorems of Green and Stokes and culminate in the connection between differential forms and cohomology theory.

Nor is this the only type of contribution that algebra and topology have made to analysis. It is, of course, a platitude to say that they have suggested new problems in analysis. A traditionalist might indeed agree that the query: "What are all the ideals in the ring of entire functions?" is admittedly a new problem for analysis, but express great disinterest in its solution. Similarly, I am sure that a fifteenth-century algebraist would have expressed an equal disdain to someone who asked about the possible nature of the set of values of a polynomial

$$w = P(z) = z + a_2z^2 + \cdots + a_mz^m,$$

for all z with $|z| < 1$. What is perhaps more convincing to a sceptic is the fact that the insights supplied by new points of view have revived interest in older classical problems, showing them to be the starting point for new attacks on fundamental questions. This has been the case recently with the problem of interpolation by bounded analytic functions, and the problem of equivalence of measure-preserving transformations.

Another theme of modern analysis that is illustrated in detail by the brilliant paper by Lorch is the role of the "abstract" approach in linear analysis. No one would deny that it is convenient to make use of a properly chosen basis at some stage in the study of a particular linear transformation. But it is almost always advantageous to postpone this step as long as possible. This is most evident when one turns to linear operators on Hilbert space; here, the use of matrices, indices, and summations becomes tedious, distracting and, indeed, misleading. Moreover, every problem seems to have its own natural basis, its own ideal set of coordinates, which is seldom the one initially given; it therefore pays to begin by looking at the space abstractly, unprejudiced by irrelevant bias.

A word or two about the specific contents of each of the papers may be in order. The lead-off paper by McShane focuses on the notion of convergence. Certainly, the idea of limit and sequence is basic to elementary calculus. However, one mark of modern analysis has been its concern with more general notions of limit, and with nonmetric topologies and phenomena that escape the restriction imposed by sequences. For example, consider the topology on functions described as "pointwise convergence."

This is nonmetrizable; if $\mathfrak{F}$ is the class of continuous functions f with $0 \leq f(x) \leq 2$ for $0 \leq x \leq 1$ and such that

$$\int_0^1 f(x)\,dx = 1,$$

then the function 2 is in the closure of $\mathfrak{F}$, although (by the Lebesgue convergence theorem) it is not the limit of any *sequence* of functions in $\mathfrak{F}$. The importance of these ideas was seen independently by E. H. Moore and M. Picone, and developed by H. L. Smith, H. Cartan, and others, emerging as a sophisticated tool formulated in terms of "nets," "directed systems," or "filters." In the present paper, McShane has given an extremely lucid introduction to these ideas, using the notion of a "direction" as the unifying principle. With many examples, he shows how a single concept of limit can be used to discuss convergence of functions, sequences, Riemann sums, and more general objects. This paper can be read by any student who has completed elementary calculus and is prepared to re-examine the notion of limit.

The second paper in the volume is reprinted, with minor changes, from the *Mathematics Magazine.* It is a justly famous paper, honored by being probably the most frequently cited research paper in history. In its original form, it is virtually unavailable; these facts alone would justify its inclusion in this volume. However, the Stone-Weierstrass theorem, as the contents of the paper have come to be known, represents one of the first and most striking examples of the success of the algebraic approach to analysis. There are many briefer proofs of the classical Weierstrass approximation theorem in the literature, but no other presentation approaches this one in its richness of insight, depth of application, and variety of structure. The subject of study is the space $\mathfrak{F}$ of continuous real-valued functions on a compact space S; the central problem is to characterize the functions that are uniform limits of functions generated from a subset $\mathfrak{F}_0$ by means of certain specified algebraic operations. In turn, Stone allows first the lattice operations max and min, then these together with addition, and finally addition and multiplication alone. In each case, the result is used to characterize the appropriate closed ideals in $\mathfrak{F}$. By specializing S and $\mathfrak{F}_0$, Stone then obtains a wide variety of interesting applications, ranging from the Tietze-Lebesgue-Urysohn extension theorem and a theorem of Dieudonné on approximation of functions of infinitely many variables, to the Peter-Weyl theorem on group representations. More unusual, perhaps, are the sections dealing with approximation on $[0, \infty)$ and $(-\infty, \infty)$, related to the study of Laguerre and Hermite polynomials. Although this paper calls for a greater mathematical sophistication than is common with undergraduates, I cannot think of a better introduction to the spirit of modern mathematics.

Turning now to the third paper, Lorch has managed, in a comparatively brief span, to give a lucid and even anecdotal account of the historical development of the spectral theorem. Starting with the three-dimensional case and a linear transformation H, he asks: "Is there a basis for the space in which the matrix for H becomes especially simple?" Motivating each transition point, the reader is led through the analysis of this case, the formulation of the general problem for symmetric operators H on Hilbert space, the nature of the solution when H is completely continuous, and the new phenomena that arise when H is allowed to be merely continuous. Finally, omitting details of the proofs but presenting the ideas convincingly, Lorch outlines a treatment of the unbounded (self-adjoint) case based upon the von Neumann approach.

The great breadth of subject-matter within modern analysis is illustrated by the fact that the concluding paper by Goffman, written independently of that by Lorch and in the same general area, has remarkably little overlap. Functional analysis—and indeed, the idea of abstract spaces themselves—seems to have arisen at the close of the nineteenth century. Initiated and encouraged by E. H. Moore and Volterra, the stimulus came in part from the study of the calculus of variations, and from recent discoveries in differential and integral equations. It is with this background that the paper by Goffman opens. Many fundamental results in classical analysis can be transcribed to assert that some continuous transformation T on a space of functions has a fixed point f_0, with $T(f_0) = f_0$; it is natural to attempt to find f_0 by examining the sequence of iterates $T^n g$, for some initial guess g. This leads in turn to the study of distance-decreasing mappings on metric spaces, and the exploration of the notion of compactness in function spaces. This again proves its usefulness when one tries to minimize a (lower semi) continuous function on an appropriate function space (space of curves, in the older terminology of Volterra). Leaving the metric case, Goffman turns to Banach and Hilbert spaces and discusses the role of normed algebras, proving in particular the Gelfand theorem on normed fields, and concluding with the algebraic proof of the theorem by Wiener on absolutely convergent Fourier series.

These articles are not intended as texts; they have more the role of commentaries, of annotated guide books to the mathematical literature. They do not attempt to bring one up to the level of current research. Their aim is preparatory, to pave the way for the more complete story that is yet to come.

They were not written for the expert; perhaps for this reason, experts will enjoy them.

A THEORY OF LIMITS

E. J. McShane

1. INTRODUCTION

One of the distinctive features of twentieth-century mathematics is its seeking for unification and generality. When two or more mathematical theories show strong resemblances, it is almost a conditioned reflex for the modern mathematician to look for the underlying common properties that cause the similarities and to construct a general theory on the basis of those common properties. The theory of limits is a good example. During the nineteenth century many limit processes were defined that led to similar theorems. In 1922 E. H. Moore and H. L. Smith† published a general theory containing the earlier theories as special cases. The fundamental theorems on limits,

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† E. H. Moore and H. L. Smith, "A general theory of limits," *American Journal of Mathematics*, vol. 44, 1922, p. 102. The same theory was devised independently by M. Picone ("Lezioni di analisi infinitesimale," *Circolo matematico di Catania*, 1923).

once proved in the general setting, could be used in all the special cases without having to be proved over and over.

In this section we shall investigate a general theory of limits that is a modification of the Moore-Smith theory; I believe that the modifications make it somewhat easier to grasp and to use.

2. NOTATION

We shall use the idea of sets and a few of the simplest relations between sets; in fact, the only symbols from set theory that we shall use are $A \subset B$ ("A is contained in B"), meaning that each member of A belongs to B; $A \cap B$ ("the intersection of A and B"), meaning the set of all things that are members of A and also are members of B; and $A \cup B$ ("the union of A and B"), meaning the set of all things belonging to A, or to B, or to both of them. Our chief interest will be in real-valued functions. But since infinite limits are too useful to reject, we augment the real number system R by adjoining two new objects, $+\infty$ and $-\infty$, and we order them by setting $-\infty < a < \infty$ for all real a. The augmented system we call the *extended real number system,* and we denote it by R^*. Since we are going to allow limits in R^*, we may as well allow functional values in R^* too.

If a and b are in R^* and $a < b$, we define the *open interval* (a, b) to be the set of all x in R^* such that $a < x < b$. Likewise, we define the *half-open intervals* $(a, b]$ and $[a, b)$ to be the sets of all x in R^* satisfying the condition $a < x \leqq b$ or $a \leqq x < b$, respectively. Also, if $a \leqq b$, the *closed interval* $[a, b]$ is the set of all x in R^* such that $a \leqq x \leqq b$. (The square bracket next to the name of either end-point indicates that the end-point is included; the round parenthesis indicates that it is not included.)

The concept of *open interval* is useful also in n-dimensional space R^n. A set J in the plane R^2 is an open interval if there exist four real numbers a, b, c, d such that $a < b$, $c < d$, and J is the set of all points (x, y) satisfying $a < x < b$, $c < y < d$. The extension to higher dimensions is obvious.

The *neighborhoods* of a point P in n-space R^n are by definition the open intervals that contain P. This takes care of R^1, or R, in particular, but not of R^*, since no open interval contains $+\infty$ or $-\infty$. In R^*, a set V is a *neighborhood* of a point b if (1) b belongs to V, and (2) V is either an open interval, a "half-line" $(c, +\infty]$ or $[-\infty, c)$, or the whole extended real-number system $R^* = [-\infty, \infty]$.

3. THE DEFINITION OF LIMIT

Let us look at two familiar definitions.

(1) *If f is defined for all real numbers, and a is real and k is in R^*, then the statement*

$$\lim_{x\to a} f(x) = k$$

means that to every neighborhood U of k there corresponds a neighborhood V of a such that whenever x is in V and $x \neq a$, $f(x)$ is in U.

(2) *If f is defined on a square S in the plane, and $P\colon(x_0, y_0)$ belongs to S, and k is in R^*, the statement*

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = k$$

means that to every neighborhood U of k there corresponds a neighborhood V of P such that whenever (x, y) is a point of S lying in V and different from P, then $f(x, y)$ is in U.

It is easy indeed to make these look alike. First, we let D be the domain of f; in (1), D is the real-number system R, and in (2) it is S. Next, each member of D will be denoted by some single letter, such as p. In (1) we mentioned a certain family of sets, each consisting of all the points of a neighborhood V of a except for a itself; in symbols, $V - \{a\}$. (Such a set is often called a *deleted neighborhood* of a.) Let us use the letter $\mathfrak{A}$ to stand for this family of sets. In (2) we used the sets of points of S belonging to V and different from P, where V is any neighborhood of P. For this example we let $\mathfrak{A}$ stand for the family of all such sets; that is, $\mathfrak{A}$ consists of all sets of the form $V \cap S - \{P\}$, V a neighborhood of P. Now both definitions take the same form:

(3) *$\lim f(p) = k$ means that to each neighborhood U of k there corresponds a set A of the family $\mathfrak{A}$ such that for all p in A, $f(p)$ is in U.*

A standard definition of the limit of a sequence $b_1, b_2, b_3, \cdots$ is the following:

(4) *The sequence $b_1, b_2, b_3, \cdots$ of real numbers has k (in R^*) as limit if for each neighborhood U of k, b_n is in U for all but finitely many values of n.*

This looks a bit less like (3). But if we let D stand for the set of positive integers, and for each positive integer p we define $f(p)$ to be another notation for a_p, and then define $\mathfrak{A}$ to be the family of all sets A each of which consists of all but finitely many of the positive integers, (4) also takes on the same form (3).

We could add other examples, but we already have enough to suggest a possible general definition of limit, as follows:

(5) *Let f be an extended-real-valued function with domain D_f, and let $\mathfrak{A}$ be any family of sets each contained in D_f. Let k be in R^*. Then the statement that $f(x)$ has k as limit (corresponding to family $\mathfrak{A}$) shall mean that whenever U is a neighborhood of k, there is a set $\mathfrak{A}$ in the family $\mathfrak{A}$ such that for every x in A, $f(x)$ is in the neighborhood U of k.*

The trouble now is that this definition of limit has such splendid generality that we are unable to prove any interesting theorems. Since we prefer to be able to prove some theorems, we put restrictions on the family $\mathfrak{A}$; of course, we try to use as few restrictions as possible. We shall in fact assume that the family $\mathfrak{A}$ in (5) has three properties, as follows:

(6) 1. *$\mathfrak{A}$ is not empty; it contains at least one set $\mathfrak{A}$.*

2. *Each set A in the family $\mathfrak{A}$ is nonempty; it contains at least one point of D_f.*

3. *If A_1 and A_2 are sets belonging to the family $\mathfrak{A}$, there is a set A_3 of the family $\mathfrak{A}$ contained in both A_1 and A_2;*

$$A_3 \subset A_1 \cap A_2.$$

In our three examples these are easily verified; in fact, in both cases the family $\mathfrak{A}$ consists of infinitely many sets each with infinitely many members, and whenever A_1 and A_2 are in $\mathfrak{A}$ so is their intersection $A_1 \cap A_2$. So these requirements are not exorbitantly strong. In fact, they are satisfied for all the classical examples, and in the last section we shall see that they cannot be weakened in any respect without allowing the possibility of undesirably weird examples. On the other hand, they are strong enough, because they allow us to prove all the traditional theorems on operations with limits.

Since we shall be repeatedly using families $\mathfrak{A}$ with properties 1, 2, and 3, it is convenient to introduce some names. A family of sets satisfying 1, 2, and 3 will be called a *direction;* if all the sets A in the family $\mathfrak{A}$ are contained in a set D, $\mathfrak{A}$ is a *direction in D*. A *directed function* is a pair consisting of (a) a function f; (b) a direction in the domain of the function. Also, if $\mathfrak{A}$ is a direction, each set A of the family $\mathfrak{A}$ will be called an *advanced set.*†

Now we can state our general definition of limit for extended-real-valued functions.

(7) *Let f be an extended-real-valued function; let $\mathfrak{A}$ be a direction in the domain of f; and let k be in R^*. The statement that the directed function $(f, \mathfrak{A})$ has k as limit (in symbols, $\lim_{x,\mathfrak{A}} f(x) = k$) is defined to mean that to each neighborhood U of k there corresponds a set A in the family $\mathfrak{A}$ such that for all x in A, $f(x)$ is in U.*

SOME FUNDAMENTAL THEOREMS. The first three theorems that we shall prove make no use of the computational properties of the number system.

(8) *Let f and g be extended-real-valued functions on the respective domains D_f and D_g, and let $\mathfrak{A}$ be a direction all of whose sets are contained both in D_f and in D_g. Assume that there is a number k in R^* such that $\lim_{x,\mathfrak{A}} f(x) = k$. If there exists a set A in $\mathfrak{A}$ such that*

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† See footnote on page 13.

$g(x) = f(x)$ for all x in A, then

$$\lim_{x,\mathfrak{A}} g(x) = k.$$

Proof: Let U be any neighborhood of k. By (7) there is a set A_1 in $\mathfrak{A}$ such that $f(x)$ is in U for all x in A. By property 1 of (6), there is a set A_2 in $\mathfrak{A}$ contained in both A and A_1. Then for all x in A_2, $f(x)$ is in U and $g(x) = f(x)$, so $g(x)$ is in U.

We now prove the important uniqueness theorem, that is, that no directed function can have more than one limit.

(9) *Let $(f, \mathfrak{A})$ be a directed function, f being extended-real-valued. If h and k are in R^*, and $h \neq k$, and $(f, \mathfrak{A})$ has k as limit, then $(f, \mathfrak{A})$ does not have h as limit.*

Proof: Let c be a number between h and k. Then k is in one of the two sets $[-\infty, c)$, $(c, \infty]$; this one we call U_1, and it is a neighborhood of k. The other we call U_2, and it is a neighborhood of h. Suppose that $(f, \mathfrak{A})$ had both h and k as limits. Since U_1 is a neighborhood of k_1, there is an A_1 in $\mathfrak{A}$ such that for all x in A_1, $f(x)$ is in U_1. Since U_2 is a neighborhood of h, there is an A_2 in $\mathfrak{A}$ such that for all x in A_2, $f(x)$ is in U_2. By property 3 of (6) there is a set A_3 in $\mathfrak{A}$ contained in both A_1 and A_2; and by property 2 of (6) there is something, say x^*, in A_3. Then x^* is in both A_1 and A_2, so $f(x^*)$ is in both U_1 and U_2. But this is impossible; U_1 and U_2 have no points in common.

(10) *If $(f, \mathfrak{A})$ is a directed function and k is in R^* and $f(x) = k$ for all x in the domain of f, then*

$$\lim_{x,\mathfrak{A}} f(x) = k.$$

Proof: Let U be any neighborhood of k. Let A_1 be any member of the family $\mathfrak{A}$; such an A_1 exists by property 1 of (6). For all x in A_1, $f(x) = k$, so $f(x)$ is in U.

4. A USEFUL VERBAL DEVICE

By a verbal device due to Halmos we can word the definitions and proofs so that the language helps to draw us along the cor-