SYMMETRY

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PREFACE
AND BIBLIOGRAPHICAL REMARKS

STARTING from the somewhat vague notion of symmetry = harmony of proportions, these four lectures gradually develop first the geometric concept of symmetry in its several forms, as bilateral, translatory, rotational, ornamental and crystallographic symmetry, etc., and finally rise to the general idea underlying all these special forms, namely that of invariance of a configuration of elements under a group of automorphic transformations. I aim at two things: on the one hand to display the great variety of applications of the principle of symmetry in the arts, in inorganic and organic nature, on the other hand to clarify step by step the philosophico-mathematical significance of the idea of symmetry. The latter purpose makes it necessary to confront the notions and theories of symmetry and relativity, while numerous illustrations supporting the text help to accomplish the former.

As readers of this book I had a wider circle in mind than that of learned specialists. It does not shun mathematics (that would defeat its purpose), but detailed treatment of most of the problems it deals with, in particular complete mathematical treatment, is beyond its scope. To the lectures, which reproduce in slightly modified version the Louis Clark Vanuxem Lectures given by the author at Princeton University in February 1951, two appendices containing mathematical proofs have been added.

Other books in the field, as for instance F. M. Jaeger's classical *Lectures on the principle* *of symmetry and its applications in natural science* (Amsterdam and London, 1917), or the much smaller and more recent booklet by Jacque Nicolle, *La symétrie et ses applications* (Paris, Albin Michel, 1950) cover only part of the material, though in a more detailed fashion. Symmetry is but a side-issue in D'Arcy Thompson's magnificent work *On growth and form* (New edition, Cambridge, Engl., and New York, 1948). Andreas Speiser's *Theorie der Gruppen von endlicher Ordnung* (3. Aufl. Berlin, 1937) and other publications by the same author are important for the synopsis of the aesthetic and mathematical aspects of the subject. Jay Hambidge's *Dynamic symmetry* (Yale University Press, 1920) has little more than the name in common with the present book. Its closest relative is perhaps the July 1949 number on symmetry of the German periodical *Studium Generale* (Vol. 2, pp. 203–278: quoted as *Studium Generale*).

A complete list of sources for the illustrations is to be found at the end of the book.

To the Princeton University Press and its editors I wish to express warm thanks for the inward and outward care they have lavished on this little volume; to the authorities of Princeton University no less sincere thanks for the opportunity they gave me to deliver this swan song on the eve of my retirement from the Institute for Advanced Study.

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If I am not mistaken the word *symmetry* is used in our everyday language in two meanings. In the one sense symmetric means something like well-proportioned, well-balanced, and symmetry denotes that sort of concordance of several parts by which they integrate into a whole. *Beauty* is bound up with symmetry. Thus Polykleitos, who wrote a book on proportion and whom the ancients praised for the harmonious perfection of his sculptures, uses the word, and Dürer follows him in setting down a canon of proportions for the human figure.[1] In this sense the idea is by no means restricted to spatial objects; the synonym "harmony" points more toward its acoustical and musical than its geometric applications. *Ebenmass* is a good German equivalent for the Greek symmetry; for like this it carries also the connotation of "middle

[1] Dürer, *Vier Bücher von menschlicher Proportion*, 1528. To be exact, Dürer himself does not use the word symmetry, but the "authorized" Latin translation by his friend Joachim Camerarius (1532) bears the title *De symmetria partium*. To Polykleitos the statement is ascribed (περὶ βελοποιϊκῶν, IV, 2) that "the employment of a great many numbers would almost engender correctness in sculpture." See also Herbert Senk, Au sujet de l'expression συμμετρία dans Diodore I, 98, 5–9, in *Chronique d'Egypte 26* (1951), pp. 63–66. Vitruvius defines: "Symmetry results from proportion . . . Proportion is the commensuration of the various constituent parts with the whole." For a more elaborate modern attempt in the same direction see George David Birkhoff, *Aesthetic measure*, Cambridge, Mass., Harvard University Press 1933, and the lectures by the same author on "A mathematical theory of aesthetics and its applications to poetry and music," *Rice Institute Pamphlet*, Vol. 19 (July, 1932), pp. 189–342.

measure," the mean toward which the virtuous should strive in their actions according to Aristotle's Nicomachean Ethics, and which Galen in *De temperamentis* describes as that state of mind which is equally removed from both extremes: σύμμετρον ὅπερ ἑκατέρου τῶν ἄκρων ἀπέχει.

The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: *bilateral symmetry*, the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is a strictly geometric and, in contrast to the vague notion of symmetry discussed before, an absolutely precise concept. A body, a spatial configuration, is symmetric with respect to a given plane E if it is carried into itself by reflection in E. Take any line l perpendicular to E and any point p on l: there exists one and only one point p' on l which has the same distance from E but lies on the other side. The point p' coincides with p only if p is on E. Reflection in E is that mapping

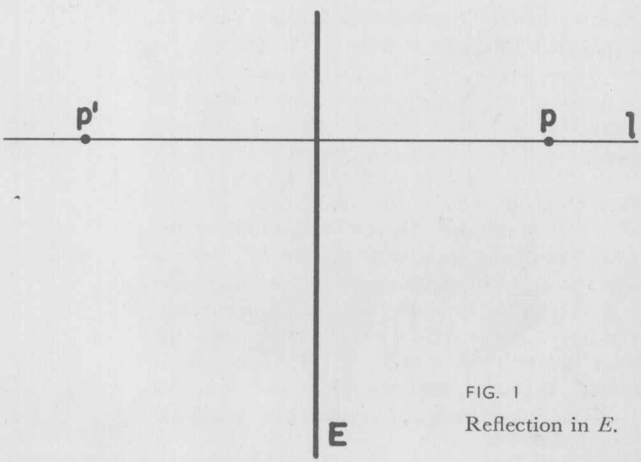


FIG. 1
Reflection in *E*.

of space upon itself, $S: p \rightarrow p'$, that carries the arbitrary point p into this its mirror image p' with respect to E. A mapping is defined whenever a rule is established by which every point p is associated with an image p'. Another example: a rotation around a perpendicular axis, say by 30°, carries each point p of space into a point p' and thus defines a mapping. A figure has rotational symmetry around an axis l if it is carried into itself by all rotations around l. Bilateral symmetry appears thus as the first case of a geometric concept of symmetry that refers to such operations as reflections or rotations. Because of their complete rotational symmetry, the circle in the plane, the sphere in space were considered by the Pythagoreans the most perfect geometric figures, and Aristotle ascribed spherical shape to the celestial bodies because any other would detract from their heavenly perfection. It is in this tradition that a modern poet[2] addresses the Divine Being as "Thou great symmetry":

God, Thou great symmetry,
Who put a biting lust in me
From whence my sorrows spring,
For all the frittered days
That I have spent in shapeless ways
Give me one perfect thing.

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty, and perfection.

The course these lectures will take is as follows. First I will discuss bilateral symmetry in some detail and its role in art as

[2] Anna Wickham, "Envoi," from *The contemplative quarry*, Harcourt, Brace and Co., 1921.

well as organic and inorganic nature. Then we shall generalize this concept gradually, in the direction indicated by our example of rotational symmetry, first staying within the confines of geometry, but then going beyond these limits through the process of mathematical abstraction along a road that will finally lead us to a mathematical idea of great generality, the Platonic idea as it were behind all the special appearances and applications of symmetry. To a certain degree this scheme is typical for all theoretic knowledge: We begin with some general but vague principle (symmetry in the first sense), then find an important case where we can give that notion a concrete precise meaning (bilateral symmetry), and from that case we gradually rise again to generality, guided more by mathematical construction and abstraction than by the mirages of philosophy; and if we are lucky we end up with an idea no less universal than the one from which we started. Gone may be much of its emotional appeal, but it has the same or even greater unifying power in the realm of thought and is exact instead of vague.

I open the discussion on bilateral symmetry by using this noble Greek sculpture from the fourth century B.C., the statue of a praying boy (Fig. 2), to let you feel as in a symbol the great significance of this type of symmetry both for life and art. One may ask whether the aesthetic value of symmetry depends on its vital value: Did the artist discover the symmetry with which nature according to some inherent law has endowed its creatures, and then copied and perfected what nature presented but in imperfect realizations; or has the aesthetic value of symmetry an independent source? I am in-

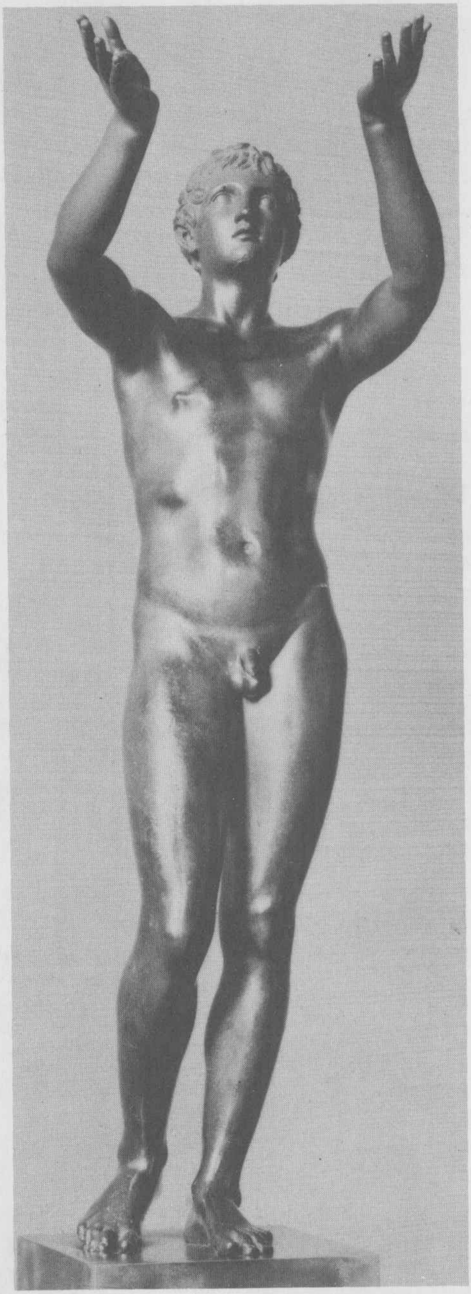
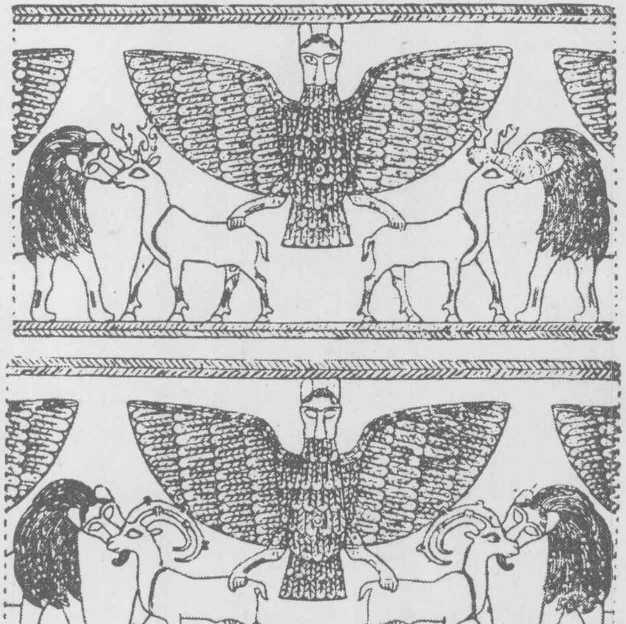


FIG. 2

clined to think with Plato that the mathematical idea is the common origin of both: the mathematical laws governing nature are the origin of symmetry in nature, the intuitive realization of the idea in the creative artist's mind its origin in art; although I am ready to admit that in the arts the fact of the bilateral symmetry of the human body in its outward appearance has acted as an additional stimulus.

Of all ancient peoples the Sumerians seem to have been particularly fond of strict bilateral or heraldic symmetry. A typical design on the famous silver vase of King Entemena, who ruled in the city of Lagash

FIG. 3



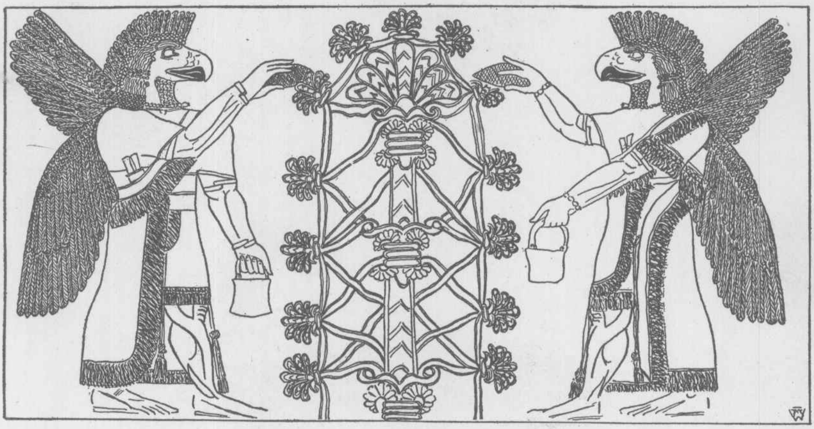


FIG. 4

around 2700 B.C., shows a lion-headed eagle with spread wings *en face*, each of whose claws grips a stag in side view, which in its turn is frontally attacked by a lion (the stags in the upper design are replaced by goats in the lower) (Fig. 3). Extension of the exact symmetry of the eagle to the other beasts obviously enforces their duplication. Not much later the eagle is given two heads facing in either direction, the formal principle of symmetry thus completely overwhelming the imitative principle of truth to nature. This heraldic design can then be followed to Persia, Syria, later to Byzantium, and anyone who lived before the First World War will remember the double-headed eagle in the coats-of-arms of Czarist Russia and the Austro-Hungarian monarchy.

Look now at this Sumerian picture (Fig. 4). The two eagle-headed men are nearly but not quite symmetric; why not? In plane geometry reflection in a vertical line l can also be brought about by rotating the plane in space around the axis l by 180°. If you look at their arms you would say these two monsters arise from each other by such rotation; the overlappings depicting their position in space prevent the plane picture from having bilateral symmetry. Yet the artist aimed at that symmetry by giving both figures a half turn toward the observer and also by the arrangement of feet and wings: the drooping wing is the right one in the left figure, the left one in the right figure.

FIG. 5

The designs on the cylindrical Babylonian seal stones are frequently ruled by heraldic symmetry. I remember seeing in the collection of my former colleague, the late Ernst Herzfeld, samples where for symmetry's sake not the head, but the lower bull-shaped part of a god's body, rendered in profile, was doubled and given four instead of two hind

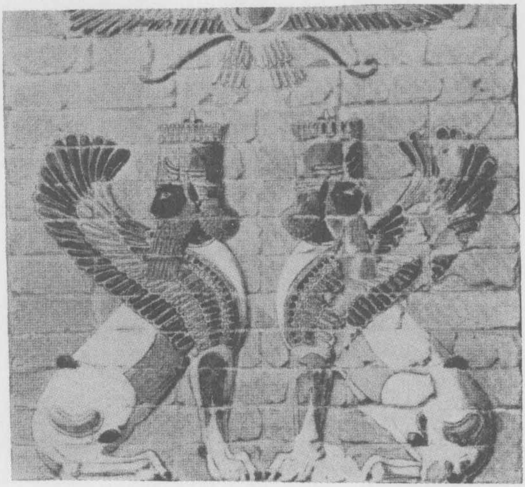


FIG. 6

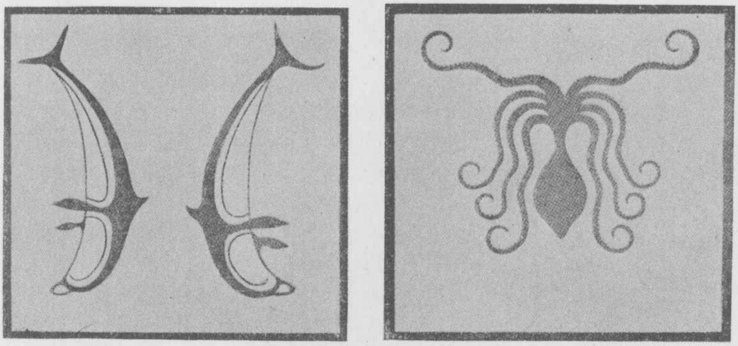


FIG. 7

legs. In Christian times one may see an analogy in certain representations of the Eucharist as on this Byzantine paten (Fig. 5), where two symmetric Christs are facing the disciples. But here symmetry is not complete and has clearly more than formal significance, for Christ on one side breaks the bread, on the other pours the wine.

Between Sumeria and Byzantium let me insert Persia: These enameled sphinxes (Fig. 6) are from Darius' palace in Susa built in the days of Marathon. Crossing the Aegean we find these floor patterns (Fig. 7) at the Megaron in Tiryns, late helladic about 1200 B.C. Who believes strongly in historic continuity and dependence will trace the graceful designs of marine life, dolphin and octopus, back to the Minoan culture of Crete, the heraldic symmetry to oriental, in the last instance Sumerian, influence. Skipping thousands of years we still see the same influences at work in this plaque (Fig. 8) from the altar enclosure in the dome of Torcello, Italy, eleventh century A.D. The peacocks drinking from a pine well among vine leaves are an ancient Christian symbol of immortality, the structural heraldic symmetry is oriental.

FIG. 8