MATHEMATICS OF CLASSICAL AND QUANTUM PHYSICS

VOLUME TWO

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PREFACE

This book is designed as a companion to the graduate level physics texts on classical mechanics, electricity, magnetism, and quantum mechanics. It grows out of a course given at Columbia University and taken by virtually all first year graduate students as a fourth basic course, thereby eliminating the need to cover this mathematical material in a piecemeal fashion within the physics courses. The two volumes into which the book is divided correspond roughly to the two semesters of the full-year course. The consolidation of the mathematics needed for graduate physics into a single course permits a unified treatment applicable to many branches of physics. At the same time the fragments of mathematical knowledge possesed by the student can be pulled together and organized in a way that is especially relevant to physics. The central unifying theme about which this book is organized is the concept of a *vector space*. To demonstrate the role of mathematics in physics, we have included numerous physical applications in the body of the text, as well as many problems of a physical nature.

Although the book is designed as a textbook to complement the basic physics courses, it aims at something more than just equipping the physicist with the mathematical techniques he needs in courses. The mathematics used in physics has changed greatly in the last forty years. It is certain to change even more rapidly during the working lifetime of physicists being educated today. Thus, the physicist must have an acquaintance with abstract mathematics if he is to keep up with his own field as the mathematical language in which it is expressed changes. It is one of the purposes of this book to introduce the physicist to the language and the style of mathematics as well as the content of those particular subjects which have contemporary relevance in physics.

The book is essentially self-contained, assuming only the standard undergraduate preparation in physics and mathematics; that is, intermediate mechanics, electricity and magnetism, introductory quantum mechanics, advanced calculus and differential equations. The level of mathematical rigor is generally comparable to that typical of mathematical texts, but not uniformly so. The degree of rigor and abstraction varies with the subject. The topics treated are of varied subtlety and mathematical sophistication, and a logical completeness that is illuminating in one topic would be tedious in another.

While it is certainly true that one does not need to be able to follow the proof of Weierstrass's theorem or the Cauchy–Goursat theorem in order to be able to compute Fourier coefficients or perform residue integrals, we feel that the student who has studied these proofs will stand a better chance of growing mathematically *after* his formal coursework has ended. No reference work, let alone a text, can cover all the mathematical results that a student will need. What *is* perhaps possible, is to generate in the student the confidence that he can find what he needs in the mathematical literature, and that he can understand it and use it. It is our aim to treat the limited number of subjects we do treat in enough detail so that after reading this book physics students will not hesitate to make direct use of the mathematical literature in their research.

The backbone of the book—the theory of vector spaces—is in Chapters 3, 4, and 5. Our presentation of this material has been greatly influenced by P. R. Halmos's text, *Finite-Dimensional Vector Spaces.* A generation of theoretical physicists has learned its vector space theory from this book. Halmos's organization of the theory of vector spaces has become so second-nature that it is impossible to acknowledge adequately his influence.

Chapters 1 and 2 are devoted primarily to the mathematics of classical physics. Chapter 1 is designed both as a review of well-known things and as an introduction of things to come. Vectors are treated in their familiar three-dimensional setting, while notation and terminology are introduced, preparing the way for subsequent generalization to abstract vectors in a vector space. In Chapter 2 we detour slightly in order to cover the mathematics of classical mechanics and develop the variational concepts which we shall use later. Chapters 3 and 4 cover the theory of finite dimensional vector spaces and operators in a way that leads, without need for subsequent revision, to infinite dimensional vector spaces (Hilbert space)—the mathematical setting of quantum mechanics. Hilbert space, the subject of Chapter 5, also provides a very convenient and unifying framework for the discussion of many of the special functions of mathematical physics. Chapter 6 on analytic function theory marks an interlude in which we establish techniques and results that are required in all branches of mathematical physics. The theme of vector spaces is interrupted in this chapter, but the relevance to physics does not diminish. Then in Chapters 7, 8, and 9 we introduce the student to several of the most important techniques of theoretical physics—the Green's function method of solving differential and partial differential equations and the theory of integral equations. Finally, in Chapter 10 we give an introduction to a subject of ever increasing importance in physics—the theory of groups.

A special effort has been made to make the problems a useful adjunct to the text. We believe that only through a concerted attack on interesting problems can a student really "learn" any subject, so we have tried to provide a large selection of problems at the end of each chapter, some illustrating or extending mathematical points, others stressing physical applications of techniques developed in the text. In the later chapters of the book, some rather significant results are left as problems or even as a programmed series of problems, on the theory that as the student develops confidence and sophistication in the early chapters he will be able, with a few hints, to obtain some nontrivial results for himself.

The text may easily be adapted for a one-semester course at the graduate (or advanced undergraduate) level by omitting certain chapters of the instructor's choosing. For example, a one-semester course could be based on Volume 1. Another possibility, and one essentially used by one of the authors at the University of California at Berkeley, is to give a semester course based on the material in Chapters 3, 4, 5, and 10. On the other hand, a one-semester course in advanced mathematical methods in physics could be constructed from Volume II.

Certain sections within a chapter which are difficult and inessential to most of the rest of the book are marked with an asterisk.

In writing a book of this kind one's debts proliferate in all directions. In addition to the book of Halmos, we have been influenced by Courant–Hilbert's treatment of, and T. D. Lee's lecture notes on, Hilbert space, Riesz and Nagy's treatment of integral equations, and M. Hamermesh's book, *Group Theory*.

A special debt of gratitude is owed to R. Friedberg whose comments on the material have been extremely helpful. In particular, the presentation of Section 5.10 is based on his lecture notes.

Parts of the manuscript have also been read and taught by Ann L. Fuller, and her comments have improved it greatly. Richard Haglund and Steven Lundeen read and commented on the manuscript. Their painstaking work has removed many blemishes, and we thank them most sincerely.

Much of this book appeared in the form of lecture notes at Columbia University. Thanks are owed to the many students there, and elsewhere, who pointed out errors, or otherwise helped to improve the manuscript. Also, the enthusiasm of the students studying this material at Berkeley provided important encouragement.

While all the above named people have helped us to improve the manuscript, we alone are responsible for the errors and inadequacies that remain. We will be grateful if readers will bring errors to our attention so corrections can be made in subsequent printings.

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CHAPTER 6

ELEMENTS AND APPLICATIONS OF THE THEORY OF ANALYTIC FUNCTIONS

INTRODUCTION

The role played by the theory of analytic functions in physics has changed considerably over the past few decades. It no longer suffices to be able to work out residue integrals; a deeper understanding of the mathematical ideas has become essential if one wants to follow current applications to physical theory. Therefore the emphasis here will be on introducing the mathematical concepts and the logical structure of the theory of analytic functions. Assuming only that the reader is familiar with the properties of complex numbers, we aim to present a self-contained account of this theory in a way that prepares one to cope with modern applications of the theory as well as those of the past.

"Imaginary" numbers were discovered in the Middle Ages in the search for a general solution of quadratic equations. It is clear from the name given them that they were regarded with suspicion. Gauss, in his doctoral thesis of 1799, gave the now familiar geometrical representation of complex numbers, and thus helped to dispel some of the mystery about them. In this century, the trend has been toward defining complex numbers as abstract symbols subject to certain formal rules of manipulation. Thus complex numbers never have taken on the "earthy" qualities of real numbers. In fact, more nearly the opposite has occurred: we have come to view real numbers abstractly as symbols obeying their own set of axioms, just like complex numbers. We now speak of *number fields*: the real field and the complex field. The axioms which define a field were stated in Chapter 3 on vector spaces.

The theory of complex numbers can be developed by viewing them as ordered pairs of real numbers, written (x, y). Let (a, b) and (c, d) be two different complex numbers, and let K be a real number. Then we define addition, multiplication of a real and a complex number, and multiplication of two complex numbers by the following rules:

1. $(a, b) + (c, d) = (a + b, c + d)$,
2. $K \cdot (a, b) = (Ka, Kb)$,
3. $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$.

From these definitions, we see that the set of all complex numbers—the complex plane—has the same mathematical structure as the set of all vectors in a plane.

This approach is followed in Landau's *Foundations of Analysis*, in which the various number systems are built up logically from Peano's five axioms; the imaginary number i is never mentioned. However, if we write the ordered pair (a, b) as $a + ib$, where $i^2 = -1$, then the above rule of complex-number multiplication is obeyed if we simply multiply out the product $(a + ib)(c + id)$ according to the usual rules of multiplication of reals. The introduction of the symbol i subsumes the ordering aspect of the ordered pair of real numbers, while extending the formal rules of arithmetic from real to complex numbers.

From the complex numbers constructed as ordered pairs of reals, where $(a, b) = a + ib$, it is possible to generalize to hypercomplex numbers of three or more components, for example $(a, b, c) = a + ib + kc$. The four-component quaternions, a type of hypercomplex number which satisfies all the rules of arithmetic except the commutative law of multiplication, are useful in dealing with rotations of a rigid body. The four 4×4 Dirac matrices, $\gamma_i (i = 1, 2, 3, 4)$, form a set of hypercomplex numbers which satisfy the anticommutative relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} .$$

It can be shown that no matter how we define addition and multiplication for these hypercomplex numbers, it is impossible to retain all the usual rules of arithmetic. As Weyl points out, the complex numbers form a natural boundary for the extension of the number concept in this respect.

6.1 ANALYTIC FUNCTIONS—THE CAUCHY-RIEMANN CONDITIONS

If to each complex number z in a certain domain there corresponds another complex number w, then w is a function of the complex variable z: $w = f(z)$. If the correspondence is one to one, we can view this as a mapping from one plane (or part of it), the z-plane, to another, the w-plane. The complex functions thus defined are equivalent to ordered pairs of real functions of two variables, because w is a complex number depending on $z = x + iy$ and therefore can be written in the form

$$w(z) = u(x, y) + iv(x, y) .$$

However, this class of functions is too general for our purposes. We are interested only in functions which are differentiable with respect to the complex variable z—a restriction which is much stronger than the condition that u and v be differentiable with respect to x and y. Therefore one of our first tasks in the study of complex function theory will be to determine the necessary and sufficient conditions for a complex function to have a derivative with respect to the complex variable z. Single-valued functions of a complex variable which have derivatives throughout a region of the complex plane are called *analytic* functions. We shall restrict our attention to this special class of complex functions.

Two examples of complex functions (both written in the form $w = u + iv$) are

1. $w = z^* = x - iy$,
2. $w = z^2 = (x + iy)^2 = x^2 - y^2 + i2xy$.

Presently, we shall show that (1) is not an analytic function, but that (2) is analytic everywhere in the complex plane; i.e., its derivative exists at all points.

Before stating exactly what is meant by the derivative of a function of a complex variable, we must have a notion of *continuity* for these functions.

In the definition that follows, mention is made of the *absolute value* of a complex number, denoted by $|z|$. The reader will recall that $|z| \equiv (zz^*)^{1/2} = (x^2 + y^2)^{1/2}$. The absolute value is sometimes called the *modulus*.

Definition. A complex function $w = f(z)$ is continuous at the point z_0 if, given any $\epsilon > 0$, there exists a δ such that $|f(z) - f(z_0)| < \epsilon$, when $|z - z_0| < \delta$, or $f(z)$ is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0) .$$

This definition is formally exactly like the definition of continuity for real functions of a real variable. However, here the absolute value signs mean that whenever z lies within a *circle* of radius δ centered at z_0 in the complex z-plane, then $f(z)$ lies within a *circle* of radius ϵ centered at $f(z_0)$ in the complex w-plane. If $f(z) = u(x, y) + iv(x, y)$, then $f(z)$ is continuous at $z_0 = x_0 + iy_0$ if u and v are continuous at (x_0, y_0).

From the class of single-valued, continuous complex functions, we now want to select those that can be differentiated. Patterning the definition of a derivative after that of real analysis, we have

Definition. $f(z)$ is differentiable at the point z_0 if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}$$

exists. We shall denote this limit, the derivative of $f(z)$ at z_0, by $f'(z_0)$.

A very important feature of the limits that occur in the definitions of continuity and the derivative is that z may approach z_0 from *any direction* on the plane. When we say the limit exists, we therefore mean that the same number must result from the limiting process regardless of how the limit is taken. This is also true in real analysis, but in that case there are only two possible directions of approach in taking the limit: from the left or the right on the real line. In real analysis, the limiting process is one-dimensional; in complex analysis, it is two-dimensional.

The equation that defines the derivative means that given any $\epsilon > 0$, there exists a δ such that

$$\left| f'(z) - \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

provided $|z - z_0| < \delta$. The requirement that the ratio $[f(z) - f(z_0)]/(z - z_0)$ always tends to the same limiting value, no matter along what path z approaches z_0, is an extremely exacting condition. The theory of analytic functions contains a number of amazing theorems, and they all result from this stringent initial requirement that the functions possess "isotropic" derivatives.

A single-valued function of z is said to be *analytic* (or *regular*) at a point z_0 if it has a derivative at z_0 and at all points in some neighborhood z_0. Thus a slight distinction is drawn between differentiability and analyticity. It pays to do this, because although there exist functions which have derivatives at certain points, or even along certain curves, no interesting results can be obtained unless functions are differentiable throughout a region, i.e., unless they are analytic. Thus if we say a function is analytic on a curve, we mean that it has a derivative at all points in a two-dimensional strip containing the curve. If a function is not analytic at a point or on a curve, we say it is *singular* there.

We shall now examine the two complex functions mentioned earlier for differentiability and analyticity. We write the derivative at z_0 in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},$$

by letting $z = z_0 + \Delta z$ in the original definition. For $f(z) = z^2$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0,$$

a result which is clearly independent of the path along which $\Delta z \to 0$, so $f(z) = z^2$ is differentiable and analytic everywhere. The result parallels exactly the result for the derivative of the real function $f(x) = x^2$.

On the other hand, if $f(z) = z^*$, we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{z_0^* + \Delta z^* - z_0^*}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z^*}{\Delta z}.$$

Now if $\Delta z \to 0$ along the real x-axis, then $\Delta z = \Delta x$ and $\Delta z^* = \Delta x^* = \Delta x$, so $f'(z_0) = +1$. However, if Δz approaches zero along the imaginary y-axis, then $\Delta z = i\Delta y$ so $\Delta z^* = -i\Delta y = -\Delta z$, so $f'(z_0) = -1$. Since at any point z_0 the limit as $z \to z_0$ depends on the direction of approach, the function is not differentiable or analytic anywhere. [As a general rule, $\Delta z^*/\Delta z = e^{-2i\theta}$, where $\theta = \tan^{-1}(\Delta y/\Delta x)$, which manifestly involves the direction of approach (θ) in taking the limit.]

Many of the theorems on differentiability in real analysis have analogs in complex analysis. For example:

1. A constant function is analytic.
2. $f(z) = z^n$ $(n = 1, 2, \cdots)$ is analytic.
3. The sum, product, or quotient of two analytic functions is analytic, provided, in the case of the quotient, that the denominator does not vanish anywhere in the region under consideration.
4. An analytic function of an analytic function is analytic.

The proofs go through exactly as in the real case.

We now determine the necessary and sufficient conditions for a function $w(z) = u(x, y) + iv(x, y)$ to be differentiable at a point. First, we assume that

$w(z)$ is in fact differentiable for some $z = z_0$. Then

$$w'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \left(\frac{\Delta u}{\Delta z} + i \frac{\Delta v}{\Delta z} \right).$$

Since $w'(z_0)$ exists, it is independent of how $\Delta z \to 0$; that is, it is independent of the ratio $\Delta y/\Delta x$. If the limit is taken along the real axis, $\Delta y = 0$, and $\Delta z = \Delta x$. Then

$$w'(z_0) = \lim_{\Delta x \to 0} \left(\frac{\Delta u}{\Delta x} + i \frac{\Delta v}{\Delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$

On the other hand, if we approach the origin along the imaginary axis, $\Delta x = 0$ and $\Delta z = i\Delta y$. Now

$$w'(z_0) = \lim_{\Delta y \to 0} \left(\frac{\Delta v}{\Delta y} - i \frac{\Delta u}{\Delta y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$

But by the assumption of differentiability, these two limits must be equal. Therefore, equating real and imaginary parts, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{6.1}$$

Equations (6.1) are known as the Cauchy-Riemann equations. They give a *necessary* condition for differentiability. We have determined this condition from special cases of the requirement of differentiability; therefore it is not surprising that these conditions alone are not sufficient.

The sufficient conditions for the differentiability of $w(z)$ at z_0 are, first, that the Cauchy-Riemann equations hold there, and second, that the first partial derivatives of $u(x, y)$ and $v(x, y)$ exist and be continuous at z_0.

The proof is straightforward. To begin, u is continuous at (x_0, y_0) because it is differentiable there; the partial derivatives of u are continuous by hypothesis. Under these assumptions, it follows from the calculus of functions of several variables* that

$$\begin{aligned} \Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) \\ &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y, \end{aligned}$$

where $\partial u/\partial x$ and $\partial u/\partial y$ are the partial derivatives evaluated at the point (x_0, y_0) and where ϵ_1 and ϵ_2 go to zero as both Δx and Δy go to zero. Using a similar formula for $v(x, y)$, we have

$$\begin{aligned} \Delta w &= w(z_0 + \Delta z) - w(z_0) = \Delta u + i\Delta v \\ &= \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y + i \left(\frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \epsilon_3 \Delta x + \epsilon_4 \Delta y \right). \end{aligned}$$

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* See, for example, G. B. Thomas, Jr., *Calculus and Analytic Geometry*, 4th Ed., Addison-Wesley Publishing Co., 1968, Section 15–4, p. 503 Eq. 4, or W. Kaplan, *Advanced Calculus*, Addison-Wesley Publishing Co., 1953, Section 2–6, p. 84.

Now using the Cauchy-Riemann equations, which by assumption hold at the point (x_0, y_0), we have

$$\Delta w = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \Delta x(\epsilon_1 + i\epsilon_3) + \Delta y(\epsilon_2 + i\epsilon_4) .$$

Therefore

$$\frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + (\epsilon_1 + i\epsilon_3)\frac{\Delta x}{\Delta z} + (\epsilon_2 + i\epsilon_4)\frac{\Delta y}{\Delta z} .$$

Since $|\Delta z| = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$, $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$, and so $|\Delta x/\Delta z| \leq 1$ and $|\Delta y/\Delta z| \leq 1$. Since these factors are bounded, the last two terms in the above equation tend to zero with Δz because ϵ_1, ϵ_2, ϵ_3, and ϵ_4 go to zero as Δz goes to zero. Therefore at z_0

$$w'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} ; \tag{6.2}$$

the limit is independent of the path followed, so the derivative exists. Using the Cauchy-Riemann conditions, we also have

$$w'(z_0) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} . \tag{6.3}$$

Example. Consider the function z^3. We have

$$z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv .$$

Thus

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial v}{\partial x} = 6xy = -\frac{\partial u}{\partial y} .$$

Thus the Cauchy-Riemann equations hold everywhere. Since the partial derivatives are continuous, the function z^3 is, in fact, analytic everywhere. A function which is analytic in the entire complex plane is said to be an *entire* function. The derivative of z^3 may be found using Eq. (6.2) or (6.3). We obtain

$$\frac{\partial z^3}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3[(x^2 - y^2) + 2ixy] = 3z^2 ,$$

a satisfying result. As a second example, we leave it to the reader to show that the function $|z|^2 \equiv zz^*$ is differentiable only at the origin, and therefore is analytic nowhere.

One remarkable result which points to connections with physics follows immediately from the Cauchy-Riemann equations. Assuming that they hold in a region, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\, \partial y} = \frac{\partial^2 v}{\partial y\, \partial x} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0 \tag{6.4}$$

if the second partial derivatives are continuous, so we can interchange the orders of differentiation in the mixed partial derivative. It follows in the same way that the function v also satisfies the two-dimensional Laplace equation. Thus both the real and imaginary parts of an analytic function with continuous second partial derivatives satisfy the two-dimensional Laplace equation. We shall later prove, using integration theory, that the second partial derivatives of an analytic function are necessarily continuous, so this qualification can be dropped. (It is interesting that these theorems about derivatives can be proved only by integration.) Any function ϕ satisfying $\nabla^2\phi = 0$ is called a *harmonic function*. If $f = u + iv$ is an analytic function, then $\nabla^2 u = \nabla^2 v = 0$, and u and v are called *conjugate* harmonic functions.

Given one of two conjugate harmonic functions, the Cauchy-Riemann equations can be used to find the other, up to a constant. For example, the function $u(x, y) = 2x - x^3 + 3xy^2$ is easily seen to be harmonic. To find its harmonic conjugate, we proceed as follows:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2 \Longrightarrow v = 2y - 3x^2y + y^3 + \phi(x) ,$$

where $\phi(x)$ is some function of x. Now, using the other Cauchy-Riemann equation, we obtain

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Longrightarrow -6xy + \phi'(x) = -6xy \Longrightarrow \phi' = 0 .$$

Thus $\phi(x)$ must be a constant, and the harmonic conjugate of u is

$$v = 2y - 3x^2y + y^3 + \text{const.}$$

Note that the function $w = u + iv = 2z - z^3 + C$ is an analytic function, as we know it must be.

Before leaving the Cauchy-Riemann conditions, let us take advantage of being physicists to present another, shorter derivation of these conditions, based on the use of infinitesimals. Let $w = u + iv$ and $w' = p + iq$. Then $\delta w = w'\delta z$, or, taking real and imaginary parts,

$$\delta u = p\delta x - q\delta y , \qquad \delta v = p\delta y + q\delta x .$$

It follows immediately that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = p , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = q .$$

These equations are identical to the Cauchy-Riemann equations (6.1).

Continuing in this informal spirit, we may derive another closely related result which provides some insight into the meaning of analyticity. Again, let $w(z) = w(x, y) = u(x, y) + iv(x, y)$. We now show that $\partial w/\partial z^* = 0$ if and only if the Cauchy-Riemann equations hold. We shall not worry about the meaning of this derivative with respect to z^*, but just differentiate formally, treating the derivative as symbolic. Using the expressions

$$x = (z + z^*)/2 \qquad \text{and} \qquad y = (z - z^*)/2i$$

we have

$$\begin{aligned}\frac{\partial w}{\partial z^*} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial z^*} \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).\end{aligned}$$

If the Cauchy-Riemann equations hold, this last expression vanishes. If, on the other hand, $\partial w/\partial z^* = 0$, then both the real and imaginary parts of the last expression must vanish, so the Cauchy-Riemann equations hold.

This purely formal result, which can be made rigorous, is trying to tell us that analytic functions are independent of z^*: they are functions of *z alone*. Thus analytic functions are true functions of a *complex* variable, not just complex functions of two real variables (see, for example, Problem 1), which will in general depend on z^* as well as z according to

$$f(x, y) = f\left(\frac{z + z^*}{2}, \frac{z - z^*}{2i}\right).$$

6.2 SOME BASIC ANALYTIC FUNCTIONS

One of the most useful functions in the complex domain is the exponential function which we define for $z = x + iy$ by

$$e^z \equiv e^x (\cos y + i \sin y) . \tag{6.5}$$

It follows easily from this definition and our earlier work that e^z is an entire function and that

$$\frac{d}{dz} e^z = e^z .$$

The other familiar properties of exponentials, in particular, $e^{z_1 + z_2} = e^{z_1} e^{z_2}$, follow readily from Eq. (6.5). We note that e^z is a periodic function of period $2\pi i$:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z (\cos 2\pi + i \sin 2\pi) = e^z .$$

From Eq. (6.5) we see that

$$e^{iy} = \cos y + i \sin y ,$$

so it follows that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2i} .$$

These relations suggest that for an arbitrary complex z we define

$$\cos z \equiv \frac{e^{iz} + e^{-iz}}{2}, \tag{6.6}$$

$$\sin z \equiv \frac{e^{iz} - e^{-iz}}{2i}. \tag{6.7}$$

Since

$$\frac{d}{dz} e^z = e^z,$$

it is a simple matter to calculate the derivatives of $\cos z$ and $\sin z$. We find that

$$\frac{d}{dz} \cos z = \frac{ie^{iz} - ie^{-iz}}{2} = -\sin z,$$

$$\frac{d}{dz} \sin z = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos z,$$

as we might expect from experience with the real variable case. Using Eqs. (6.6) and (6.7), it is a simple matter to verify that all the familiar trigonometric identities, such as

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

continue to be valid for complex variables.

The complex functions sine and cosine may, of course, be put in the form $u(x, y) + iv(x, y)$. For example,

$$\begin{aligned} \sin z &= \frac{1}{2i}[e^{i(x+iy)} - e^{-i(x+iy)}] \\ &= \frac{1}{2i} e^{-y}(\cos x + i \sin x) - \frac{1}{2i} e^{y}(\cos x - i \sin x) \\ &= \sin x(e^y + e^{-y})/2 + i \cos x(e^y - e^{-y})/2. \end{aligned}$$

Therefore

$$\sin z = \cosh y \sin x + i \sinh y \cos x. \tag{6.8}$$

Similarly,

$$\cos z = \cosh y \cos x - i \sinh y \sin x. \tag{6.9}$$

Setting $x = 0$, we obtain the useful relations $\sin(iy) = i \sinh y$ and $\cos(iy) = \cosh y$. We also see that the Cauchy-Riemann conditions are satisfied everywhere, as we know they must be. Other properties which follow directly from Eqs. (6.8) and (6.9) are

$$(\sin z)^* = \sin(z^*),$$

$$\sin(-z) = -\sin(z),$$

$$\sin(z + 2\pi) = \sin(z).$$

Using the sine and cosine functions, we can define the other familiar trigonometric functions. For example,

$$\tan z = \sin z/\cos z\ ;$$

similar extensions of the real case are defined for the cotangent, secant, and cosecant. These functions differ from the sine and cosine in that they are *not* analytic everywhere. The tangent, being the ratio of two analytic functions, will be analytic everywhere *except* at points where $\cos z = 0$. Using the real and imaginary parts of the cosine, we can rewrite this condition as

$$\cosh y \cos x = 0\ , \qquad \sinh y \sin x = 0\ .$$

Now $\cosh y \geq 1$ for all real y, so the first equation has a solution whenever $\cos x = 0$, or $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \cdots$. At these points, $\sin x = \pm 1$, so the second equation requires that $\sinh y = 0$, that is, $y = 0$. Thus the tangent function is singular at the points $(2n + 1)\pi/2$, $(n = 0, \pm 1, \cdots)$ on the real axis, and only at these points. Therefore $\tan z$ becomes infinite at precisely those points where $\tan x$ (real x) becomes infinite and *only* at those points.

On the basis of the above discussion, one might be tempted to think that the complex trigonometric functions are "just the same thing" as their real counterparts. However, the reader can easily show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y\ ,$$

and this expression increases *without limit* as y tends to infinity. This is in marked contrast with the real case, where $|\sin x| \leq 1$ for all real x.

The functions which we have discussed thus far all have the property that if we pick any point z_0 in the complex plane and follow any path from z_0 through the plane back to z_0, then the value of the function changes continuously along the path, returning to its original value at z_0. For example, suppose that we consider the function $w(z) = e^z$ and start at the point $z_0 = 1$, encircling the origin in the z-plane counterclockwise along the unit circle. Figure 6.1(a) shows the circular path in the z-plane, and Fig. 6.1(b) shows the corresponding path in the w-plane. [The use of two complex planes to "graph" the function $w(z)$ is often employed in complex variable theory.] We note that both paths are closed, which is just the geometrical statement of the fact that if we start at a point z_0, where the function has the value $w(z_0)$, then when we move along a closed curve back to z_0, the functional values also follow a smooth path back to $w(z_0)$.

Now for e^z this result is hardly surprising since we have *defined* e^z in such a way as to ensure this behavior, letting ourselves be guided by the properties of the real exponential function. Now if we look at another simple function, namely, the square root, we see that things do not always go so smoothly. Let us write formally

$$w(z) \equiv \sqrt{z} \equiv \sqrt{x + iy}\ .$$