Victor Isakov

Inverse Problems for Partial Differential Equations

偏微分方程逆问题

Applied Mathematical Sciences 127

Springer
世界图书出版公司

Victor Isakov

Inverse Problems for Partial Differential Equations

Springer

世界图书出版公司

书　　名：Inverse Problems for Partial Differential Equations
作　　者：Victor Isakov
中 译 名：偏微分方程用的逆问题
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：010-64015659, 64038347
电子信箱：kjsk@vip.sina.com
开　　本：24 开　　　印　张：12.5
出版年代：2004 年 4 月
书　　号：7-5062-6612-1/O·465
版权登记：图字：01-2004-1159
定　　价：59.00 元

Victor Isakov
Department of Mathematics and Statistics
The Wichita State University
Wichita, KS 67260-0033
USA

Editors

Mathematics Subject Classification (1991): 35R30, 86A22, 80A23, 65M30, 35JXX, 35L

With four illustrations.

Library of Congress Cataloging-in-Publication Data
Isakov, Victor, 1947–
Inverse problems for partial differential equations / Victor Isakov.
p. cm. – (Applied mathematical sciences ; 127)
Includes bibliographical references (p. –) and index.
ISBN 0-387-98256-6 (alk. paper)
1. Inverse problems (Differential equations) 2. Differential equations, Partial. I. Title. II. Series: Applied mathematical sciences (Springer-Verlag New York Inc.) ; v. 127.
QA1.A647 vol. 127
[QA374]
510 s–dc21 97-16663
[515'.353]

Reprinted in China by Beijing World Publishing Corporation, 2004

9 8 7 6 5 4 3 2 1

ISBN 0-387-98256-6 Springer-Verlag New York Berlin Heidelberg SPIN 10628737

Preface

This book describes the contemporary state of the theory and some numerical aspects of inverse problems in partial differential equations. The topic is of substantial and growing interest for many scientists and engineers, and accordingly to graduate students in these areas. Mathematically, these problems are relatively new and quite challenging due to the lack of conventional stability and to nonlinearity and nonconvexity. Applications include recovery of inclusions from anomalies of their gravitational fields; reconstruction of the interior of the human body from exterior electrical, ultrasonic, and magnetic measurements, recovery of interior structural parameters of detail of machines and of the underground from similar data (non-destructive evaluation); and locating flying or navigated objects from their acoustic or electromagnetic fields. Currently, there are hundreds of publications containing new and interesting results. A purpose of the book is to collect and present many of them in a readable and informative form. Rigorous proofs are presented whenever they are relatively short and can be demonstrated by quite general mathematical techniques. Also, we prefer to present results that from our point of view contain fresh and promising ideas. In some cases there is no complete mathematical theory, so we give only available results. We do not assume that a reader possesses an enormous mathematical technique. In fact, a moderate knowledge of partial differential equations, of the Fourier transform, and of basic functional analysis will suffice. However, some details of proofs need quite special and sophisticated methods, but we hope that even without completely understanding these details a reader will find considerable useful and stimulating material. Moreover, we start many chapters with general information about the direct problem, where we collect, in the form of theorems, known (but not simple and not always easy to find) results that are needed in the treatment of inverse problems. We hope that this book (or at least most of it) can be used as a graduate text. Not only do we present recent achievements, but we formulate basic inverse problems, discuss regularization, give a short review of uniqueness in the Cauchy problem, and include several exercises that sometimes substantially complement the book. All of them can be solved by using some modification of the presented methods.

Parts of the book in a preliminary form have been presented as graduate courses at the Johannes-Kepler University of Linz, at the University of Kyoto, and at

Wichita State University. Many exercises have been solved by students, while most of the research problems await solutions. Parts of the final version of the manuscript have been read by Ilya Bushuyev, Alan Elcrat, Matthias Eller, and Peter Kuchment, who found several misprints and suggested many corrections. The author is grateful to these colleagues for their attention and help. He also thanks the National Science Foundation for long-term support of his research, which stimulated the writing of this book.

Wichita, Kansas Victor Isakov

Contents

1

Inverse Problems

In this chapter we formulate basic inverse problems and indicate their applications. The choice of these problems is not random. We think that it represents their interconnections and some hierarchy.

An inverse problem assumes a direct problem that is a well-posed problem of mathematical physics. In other words, if we know completely a "physical device," we have a classical mathematical description of this device including uniqueness, stability, and existence of a solution of the corresponding mathematical problem. But if one of the (functional) parameters describing this device is to be found (from additional boundary/experimental) data, then we arrive at an inverse problem.

1.1 The inverse problem of gravimetry

The gravitational field u, which can be measured and perceived by the gravitational force ∇u and which is generated by the mass distribution f, is a solution to the Poisson equation

$$-\Delta u = f \tag{1.1.1}$$

in $\mathbb{R}^3$, where $\lim u(x) = 0$ as $|x|$ goes to $+\infty$. For modeling and for computational reasons, it is useful to consider as well the plane case ($\mathbb{R}^2$). Then the behavior at infinity must be $u(x) = C ln|x| + u_0(x)$, where u_0 goes to zero at infinity. One assumes that f is zero outside a finite domain Ω, which is a ball or a body close to a ball (earth) in gravimetry. The direct problem of gravimetry is to find u given f. This is a well-posed problem: Its solution exists for any integrable f, and even for any distribution that is zero outside Ω; it is unique and stable with respect to standard functional spaces. As a result, the boundary value problem (1.1.1) can be solved numerically by using difference schemes, although these computations are not very easy in the three-dimensional case. This solution is given by the Newtonian potential

$$u(x) = \int_\Omega k(x-y)f(y)dy, \qquad k(x) = 1/(4\pi|x|) \tag{1.1.2}$$

(or $k(x) = -1/(2\pi)\ln|x|$ in $\mathbb{R}^2$). Practically we perceive and can measure only the gravitational force ∇u.

The *inverse problem of gravimetry* is to find f given ∇u on Γ, which is a part of the boundary $\partial\Omega$, of Ω.

This problem was actually formulated by Laplace, but the first (and simplest) results were obtained only by Stokes in the 1860s and Herglotz about 1910 [Her]. We will analyze this problem in Sections 2.1–2.2 and 4.1. There is an advanced mathematical theory of this problem presented in a book of the author [Is4]. It is fundamental in geophysics, since it simulates recovery of the interior of the earth from boundary measurements of the gravitational field. Unfortunately, there is a strong nonuniqueness of f for a given gravitational potential outside Ω. However, if we look for a more special type of f (like harmonic functions, functions independent of one variable, or characteristic functions $\chi(D)$ of unknown domains D inside Ω), then there is uniqueness, and f can be recovered from u given outside Ω, theoretically and numerically. In particular, one can show uniqueness of $f = \chi(D)$ when D is either star-shaped with respect to its center of gravity or convex with respect to one of the coordinates.

An important feature of the inverse problem of gravimetry is its ill-posedness, which creates many mathematical difficulties (absence of existence theorems due to the fact that ranges of operators of this problem are not closed in classical functional spaces) and numerical difficulties (stability under constraints is (logarithmically) weak, and therefore convergence of iterative algorithms is very slow, so numerical errors accumulate and do not allow good resolution). In fact, it was Tikhonov who in 1944 observed that introduction of constraints can restore some stability to this problem, and this observation was one of starting points of the contemporary theory of ill-posed problems.

This problem is fundamental in recovering the density of the earth by interpreting results of measurements of the gravitational field (gravitational anomalies). Another interesting application is in gravitational navigation. One can measure the gravitational field (from satellites) with quite high precision, and then possibly find the function f that produces this field, and then use these results to navigate aircraft. The point is that to navigate aircraft one needs to know u near the surface of the earth Ω, and finding f supported in $\bar{\Omega}$ gives u everywhere outside of Ω by solving a much easier direct problem of gravimetry. The advantage of this method is that the gravitational field is the most stationary and stable of all known physical fields, so it can serve in navigation. The inverse problem here is a way to code and store information about the gravitational field. This problem is quite unstable, but still manegeable. We discuss this problem in Sections 2.2, 2.3, 3.3, 4.1, and in Chapter 10. The direct problem is much easier: given f, one can find the field u by precise, stable, and effective numerical procedures.

Inverse gravimetry is a classical example of an inverse source problem, where one is looking for the right side of a differential equation (or a system of equations) from extra boundary data. Consider a simple example: in the second-order ordinary differential equation $-u'' = f$ on $\mathbb{R}$, let Ω be the interval $(-1, 1)$. Let $u_0 = u(-1)$,

$u_1 = u'(-1)$; then

$$u(x) = u_0 + u_1(x+1) - \int_{-1}^{x} (x-y)f(y)dy \text{ when } -1 < x < 1.$$

The prescrition of the Cauchy data u, u' at $t = 1$ is equivalent to the prescription of two integrals

$$\int_{\Omega} (1-y)f(y)dy \text{ and } \int_{\Omega} f(y)dy.$$

We cannot determine more given the Cauchy data at $t = -1, 1$, no matter what the original Cauchy data. The same information about f is obtained if we prescribe any u on $\partial\Omega$ and if in addition we know u'. In particular, nonuniqueness is substantial: one cannot find a function from two numbers. If we add to f any function f_0 such that

$$\int_{\Omega} v(y) f_0(y)dy = 0$$

for any linear function v (i.e., for any solution of the adjoint equation $-v'' = 0$), then according to the above formulae we will not change the Cauchy data on $\partial\Omega$. The situation with partial differential equations is quite similar, although more complicated.

If ∇u is given on Γ, then u can be found uniquely outside Ω by uniqueness in the Cauchy problem for harmonic functions and the assumptions on the behavior at infinity. Observe that given u on $\partial\Omega \subset \mathbb{R}^3$ one can solve the exterior Dirichlet problem for u outside Ω and find $\partial_\nu u$ on $\partial\Omega$, so in fact we are given the Cauchy data there.

Exercise 1.1.1. Assume that Ω is the unit disk $\{|x| < 1\}$ in $\mathbb{R}^2$. Show that a solution f of the inverse gravimetry problem that satisfies one of the following three conditions is unique. (1) It does not depend on $r = |x|$. (2) it satisfies the second-order equation $\partial_2^2 f = 0$. (3) It satisfies the Laplace Equation $\Delta f = 0$ in Ω. In fact, in the cases (2) and (3), Ω can be any bounded domain with $\partial\Omega \in C^3$ with connected $\mathbb{R}^2 \backslash \bar{\Omega}$. {*Hint*: to handle case (1) consider $v = r\partial_r u$ and observe that v is harmonic in Ω. Determine v in Ω by solving the Dirichlet problem and then find f. In cases (2) and (3) introduce new unknown (harmonic in Ω) functions $v = \partial_2^2 u$ and $v = \Delta u$.}

Exercise 1.1.2. In the situation of Exercise 1.1.1 prove that a density $f(r)$ creates zero exterior potential if and only if

$$\int_0^1 r f(r)dr = 0.$$

{*Hint*: make use of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and of the expression for the Laplacian in polar coordinates,

$$\Delta = r^{-1}(\partial_r(r\partial_r) + \partial_\theta(r^{-1}\partial_\theta)).$$

Observe that for such f the potential u does not depend on θ, and perform an analysis similar to that given above for the simplest differential equation of second order.}

What we discussed briefly above can be called the density problem. It is linear with respect to f. The domain problem when one is looking for the unknown D is apparently nonlinear and seems (and indeed is) more difficult. In this introduction we simply illustrate it by recalling that the Newtonian potential U of the ball $D = B(a; R)$ of constant density ρ is given by the formulae

(1.1.3)
$$U(x; \rho\chi(B(a; R))) = \begin{bmatrix} R^3\rho/3|x-a|^{-1} & \text{when } |x-a| \geq R; \\ R^2\rho/2 - \rho/6|x-a|^2 & \text{when } |x-a| < R. \end{bmatrix}$$

These formulae imply that a ball and its constant density cannot be simultaneously determined by their exterior potential ($|x - a| > R$). One can only find $R^3\rho$. Moreover, according to (1.1.2) and (1.1.3), the exterior Newtonian potential of the annulus $A(a; R_1, R_2) = B(a; R_2)\backslash \bar{B}(a; R_1)$ is $(R_2^3 - R_1^3)\rho/3|x - a|^2$, so only $\rho(R_2^3 - R_1^3)$ can be found. In fact, in this example the cavity of an annulus further deteriorates uniqueness. The formulae (1.1.3) can be obtained by observing the rotational (around a) invariancy of the equation (1.1.1) when $f = \rho\chi(B(a; R))$ and using this equation in polar coordinates together with the continuity of the potential and first order derivatives of the potential at ∂D.

We will give more detail on this interesting and not completely resolved problem in Section 4.1, observing that starting from the pioneering work of P. Novikov [No], uniqueness and stability results have been obtained by Prilepko [Pr], Sretensky, and the author [Is4].

There is another interesting problem in geophysics, that of finding the shape of the geoid D given the gravitational potential at its surface. Mathematically, like the domain problem in gravimetry, it is a free boundary problem that consists in finding a bounded domain D and a function u satisfying the conditions

$$\Delta u = \rho \text{ in } D, \qquad \Delta u = 0 \text{ outside } \bar{D},$$
$$u, \nabla u \in C(\mathbb{R}^3), \quad \lim u(x) = 0 \text{ as } |x| \to \infty,$$
$$u = g \text{ on } \partial D,$$

where g is a given function. To specify the boundary condition, we assume that D is star-shaped, so it is given in polar coordinates (r, σ) by the equation $r < d(\sigma)$, $|\sigma| = 1$. Then the boundary condition should be understood as $u(d(\sigma)\sigma) = g(\sigma)$, where g is a given function on the unit sphere. This problem is called the Molodensky problem, and it was the subject of recent intensive study by both mathematicians and geophysicists. Again, despite certain progress, there are many challenging questions, in particular, the global uniqueness of a solution is not known.

To describe electrical and magnetic phenomena one makes use of single- and double-layer potentials

$$U^{(1)}(x; gd\Gamma) = \int_\Gamma K(x, y)g(y)d\Gamma(y)$$

and

$$U^{(2)}(x; gd\Gamma) = \int_\Gamma \partial_{\nu(y)}K(x, y)g(y)d\Gamma(y)$$

distributed with (measurable and bounded) density g over a piecewise-smooth bounded surface Γ in $\mathbb{R}^3$. As in inverse gravimetry, one is looking for g and Γ (or for one of them) given one of these potentials outside a reference domain Ω. The inverse problem for the single-layer potential can be used, for example, in gravitational navigation: it is probably more efficient to look for a single layer distribution g instead of the volume distribution f. As a good example of a practically important problem about double layer potentials we mention that of exploring the human brain to find active parts of its surface Γ_c (cortical surface). The active parts occupy not more than ten percent of Γ_c. They produce a magnetic field that can be described as the double-layer potential distributed over Γ_c with density $g(y)$, and one can (quite precisely) measure this field outside the head Ω of the patient. We have the integral equation of the first kind

$$G(x) = \int_{\Gamma_c} \partial_{\nu(y)}K(x, y)g(y)d\Gamma(y), \quad x \in \partial\Omega,$$

where Γ_c is a given C^1-surface, $\bar{\Gamma}_c \subset \Omega$, and $g \in L^\infty(\Gamma_c)$ is an unknown function. In addition to its obvious ill-posedness, an intrinsic feature of this problem is the complicated shape of Γ_c. There have been only preliminary attempts to solve it numerically. No doubt a rigorous mathematical analysis of the problem (asymptotic formulae for the double-layer potential when Γ_c is replaced by a closed smooth surface or, say, use of homogenization) could help a lot.

In fact, it is not very difficult to prove uniqueness of g (up to a constant) with the given exterior potential of the double layer.

We observe that in inverse source problems one is looking for a function f of the partial differential equation $-\Delta u = f$ when its solution u is known outside Ω. If one allows f to be a measure or a distribution of first order, then the inverse problems about the density g of a single or double layer can be considered as an inverse source problem with $f = d\Gamma$ or $f = g\partial_\nu(d\Gamma)$.

1.2 The inverse conductivity problem

The conductivity equation for electric (voltage) potential u is

(1.2.1) $$\operatorname{div}(a\nabla u) = 0 \text{ in } \Omega.$$

For a unique determination of u one can prescribe at the boundary the Dirichlet data

$$u = g \text{ on } \partial\Omega. \tag{1.2.2}$$

Here we assume that a is a scalar function, $\epsilon_0 \leq a$, that is measurable and bounded. In this case one can show that there is a unique solution $u \in H_{(1)}(\Omega)$ to the direct problem (1.2.1)–(1.2.2), provided that $g \in H_{(1/2)}(\partial\Omega)$ and $\partial\Omega$ is Lipschitz. Moreover, there is stability of u with respect to g in the norms of these spaces. In other words, we have the well-posed direct problem.

Often we can assume that a is constant near $\partial\Omega$. Then, if $g \in C^2(\partial\Omega)$, the solution $u \in C^1$ near $\partial\Omega$, so the following classical Neumann data are well-defined:

$$a\partial_\nu u = h \quad \text{on } \Gamma, \tag{1.2.3}$$

where Γ is a part of $\partial\Omega \in C^2$.

The *inverse conductivity problem* is to find a given h for one g (one boundary measurement) or for all g (many boundary measurements).

In many applied situations it is h that is prescribed on $\partial\Omega$ and g that is measured on Γ. This makes some difference (not significant theoretically and computationally) in the case of single boundary measurements but makes almost no difference in the case of many boundary measurements when $\Gamma = \partial\Omega$, since actually it is the set of Cauchy data $\{g, h\}$ that is given. The study of this problem was initiated by Langer [La] as early as the 1930s.

The inverse conductivity problem looks more difficult than the inverse gravimetric one: it is "more nonlinear." On the other hand, since u is the factor of a in the equation (1.2.1), one can anticipate that many boundary measurements provide much more information about a than one boundary measurement. We will show later that this is true when the dimension $n \geq 2$. When $n = 1$, the amount of information about a from one or many boundary measurements is almost the same.

This problem lays a mathematical foundation to electrical impedance tomography, which is a new and promising method of prospecting the interior of the human body by surface electromagnetic measurements. On the surface one prescribes current sources (like electrodes) and measures voltage (or vice versa) for some or all positions of those sources. The same mathematical model works in a variety of applications, such as magnetometric methods in geophysics, mine and rock detection, and the search for underground water.

In the following exercise it is advisable to use polar coordinates (r, θ) in the plane and separation of variables.

Exercise 1.2.1. Consider the inverse conductivity problem for $\Omega = \{r < 1\}$ in $\mathbb{R}^2$ with many boundary measurements when $a(x) = a(r)$. Show that this problem is equivalent to the determination of a from the sequence of the Neumann data $w_k'(1)$ of the solutions to the ordinary differential equations $-r(arw')' - k^2 aw = 0$ on $(0, 1)$ bounded at $r = 0$ and satisfying the boundary condition $w(1) = 1$.

We will conclude this section with a discussion of the origins of equation (1.2.1), which we hope will illuminate possible applications of the inverse conductivity problem.

The first source is in Maxwell's system for electromagnetic waves of frequency ω:

$$\begin{aligned} \operatorname{curl} E &= -i\omega\mu H, \\ \operatorname{curl} H &= \sigma E + i\omega\epsilon E, \end{aligned} \tag{1.2.4}$$

where E, H are electric and magnetic vectors and σ, ϵ, and μ are respectively conductivity, electric permittivity, and magnetic permeability of the medium. In the human body μ is small, so we neglect it and conclude that *curl* $E = 0$ in Ω. Assuming that this domain is simply connected, we can state that E is a potential field; i.e., $E = \nabla u$. Since it is always true that div *curl* $H = 0$, we obtain for u equation (1.2.1) with

$$a = \sigma + i\omega\epsilon. \tag{1.2.5}$$

Observe that in medical applications σ and ϵ are positive functions of x and ω. In certain important situations one can assume that ϵ is small and therefore obtain equation (1.2.1) with the real-valued coefficient $a = \sigma$, which is to be found from exterior boundary measurements. This explains what the problem has to do with inverse conductivity. An important feature of the human body is that conductivites of various regions occupied by basic components are known constants, and actually one is looking for the shapes of these regions. For example, conductivities of muscles, lungs, bones, and blood are respectively 8.0, 1.0, 0.06, and 6.7.

In geophysics the same equation is used to describe prospecting by use of magnetic fields. Moreover, it is a steady-state equation for the temperature u. Indeed, if at the boundary of a domain Ω we maintain time-independent temperature $g(x)$, $x \in \partial\Omega$, then (Section 9.0) a solution of the heat equation $\partial_t U = \operatorname{div}(a\nabla U)$ in Ω, $0 < t$, is (exponentially) rapidly convergent to a steady-state solution u to the equation (1.2.1) with the Dirichlet boundary condition (1.2.2). The function a then is called the thermal conductivity of the medium and is to be found in several engineering applications.

So, the inverse conductivity problem applies to a variety of situations when important interior characteristics of a physical body are to be found from boundary experiments and observations of fundamental fields.

1.3 Inverse scattering

In inverse scattering one is looking for an object (an obstacle D or a medium parameter) from results of observations of so-called field generated by (plane) incident waves of frequency k. The field itself (acoustic, electromagnetic, or elastic) in the simplest situation of scattering by an obstacle D is a solution u to the

Helmholtz equation

(1.3.1) $$-\Delta u - k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \bar{D}$$

satisfying the homogeneous Dirichlet boundary condition

(1.3.2) $$u = 0 \text{ on } \partial D \quad \text{(soft obstacle)}$$

or another boundary condition, like the Neumann condition

($1.3.2_h$) $$\partial_\nu u + bu = 0 \text{ on } \partial D \text{ (hard obstacle).}$$

This solution is assumed to be the sum of the plane incident wave u^i and a scattered wave u^s that is due to the presence of an obstacle

(1.3.3) $$u(x) = u^i(x) + u^s(x),$$

where $u^i = \exp(ik\xi \cdot x)$. In some situations spherical incident waves u^i (depending only on $|x|$) are more useful and natural.

Basic examples of scattering by a medium are obtained when one replaces equation (1.3.1) by the equation

(1.3.4) $$-\Delta u + (c - ikb_0 - k^2 a_0)u = 0 \quad \text{in } \mathbb{R}^3.$$

The coefficients a_0, b_0, c are assumed to be in $L^\infty(\Omega)$ for a bounded domain Ω, with b_0, c zero outside Ω and $a_0 > \epsilon > 0$ and equal to 1 outside Ω. In the representation (1.3.3) the first term in the right side is a simplest solution of the Helmholtz equation in $\mathbb{R}^3$ when there is no obstacle or perturbation of coefficients. In the presence of obstacles the solutions are different, and the additional term u^s can be interpreted as a wave scattered from an obstacle or perturbation.

It can be shown that for any incident direction $\xi \in \Sigma$ there is a unique solution u of the scattering problem (1.3.1), (1.3.2), (1.3.3) or (1.3.3), (1.3.4), where the scattered field satisfies the Sommerfeld radiation condition

(1.3.5) $$\lim r(\partial_r u^s - iku^s) = 0 \text{ as } r \text{ goes to } +\infty.$$

This condition guarantees that the wave $u(x)$ is an outgoing one. For soft obstacles one can assume $\partial D \in C^2$, $\mathbb{R}^3 \setminus \bar{D}$ is connected, and then $u \in C^1(\mathbb{R}^3 \setminus D)$. For scattering by medium we have $u \in C^1$ and $u \in C^2$ when $c, b_0, a_0 \in C^l$. We discuss solvability in more detail in Chapter 6.

Any solution to the Helmholtz equation outside of Ω that satisfies condition (1.3.5) admits the representation

(1.3.6) $$u^s(x) = \exp(ikr)/r A(\sigma, \xi; k) + O(r^{-2}),$$

where A is called the *scattering amplitude*, or *far field pattern*.

The *inverse scattering problem* is to find a scatterer (obstacle or medium) from far field pattern.

This problem is fundamental for exploring bodies by acoustic or electromagnetic waves. The inverse medium problem with $a_0 = 1$, $b_0 = 0$ is basic in quantum mechanics, as suggested by Schrödinger in the 1930s because quantum mechanical