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Linear Canonical Transforms

Theory and Applications

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Foreword

Waves underlie a wealth of natural phenomena, ranging from seismic activity to elementary particles, and encompassing light and sound. Mathematical tools that are useful for modeling and understanding the behavior of waves are therefore of central importance in science and engineering. The insights of Huygens and Fresnel led to the description of wave propagation in terms of secondary waves: if a wave field is known at a given initial plane, its propagation away from it can be modeled by expressing the field as a continuous superposition of secondary waves emanating from all points over the plane. This beautiful interpretation provides a connection between wave propagation and linear integral transformations. As underlined by Feynman's path-integral formalism of quantum mechanics, this interpretation also holds for the description of the temporal evolution of quantum-mechanical wave functions, where time plays the role of the propagation direction, and instead of an initial plane one must consider all space at an initial time.

The mathematical similarity between different wave phenomena becomes more accentuated when regimes that allow certain approximations are considered. For example, in the description of optical waves of a given temporal frequency, one is often interested in highly collimated beams that propagate mainly around a specific direction. In this case, the electric field distribution satisfies approximately what is known as the paraxial wave equation. This equation is mathematically similar to the Schrödinger equation ruling the evolution of quantum wave functions in the nonrelativistic regime. Further, the effect of some refractive index distributions on the propagation of an optical field can be formally analogous to that of some potentials over the evolution of a particle's wave function. It is then natural that the same propagation models be employed in the description of these systems.

This book gives a thorough overview of a class of integral transformations, known as linear canonical transformations, which are remarkable both for their mathematical elegance and for their range of physical applications. Mathematically, linear canonical transformations are defined by their simple properties: (1) each of these transformations is associated with, and fully determined by, a 2×2 matrix (or $2N \times 2N$ matrix, when applied to functions that depend on N variables); (2) a concatenation of a series of linear canonical transformations can be reduced to a single linear canonical transformation whose matrix is the product of the matrices for the original independent transformations. Physically, linear canonical transformations describe wave propagation in cases where the Hamiltonian is at most quadratic in both position (e.g., thin lenses and quadratic gradient index media in optics, or harmonic-oscillator potentials in quantum mechanics) and momentum (i.e., within the paraxial approximation in optics or the non-relativistic approximation in quantum mechanics). In these contexts, the matrix associated with the transformation turns out to be the transfer matrix that maps the initial position and momentum of a classical particle or ray to the final ones for the system in question. Linear canonical transformations include as special cases the Fourier transformation, the fractional Fourier transformation (which describes the paraxial propagation of optical fields in quadratic gradient index fibers, as well as the evolution of quantum states in a harmonic oscillator potential), the Fresnel transformation (which describes free propagation of paraxial wave beams), and even simple multiplication by quadratic phase factors.

This book is, to my knowledge, the first devoted fully to providing a comprehensive study of linear canonical transformations and their applications. Some previous publications have included some discussions on these transformations, while others have focused on specific special cases like the Fourier or even the fractional Fourier transformations. While some of these special cases are standard items in the toolbox of most physicists and engineers, the more general class of transformations discussed here is not as widely known. The present book is therefore a very timely and welcome addition to the scientific literature. Further, its chapters are authored by some of the most influential researchers in the subject. The first part of the book concentrates on the origins, definition, and properties of linear canonical transformations. Chapter 1, by Kurt Bernardo Wolf, gives a historical perspective on the independent development of linear canonical transformations in optics and nuclear physics, from the point of view of someone at the intersection of these two communities. In Chap. 2, Martin J. Bastiaans and Tatiana Alieva provide a detailed treatment of the definition and properties of linear canonical transformations, paying careful attention to cases of special interest. The eigenfunctions of linear canonical transformations, i.e. those functions that retain their functional form following transformation, are discussed by Soo-Chang Pei and Jian-Jiun Ding in Chap. 3. The different types of uncertainty relations between functions and their linear canonical transforms are the subject of Chap. 4, by R. Tao. In Chap. 5, Tatiana Alieva, José A. Rodrigo, Alejandro Cámara, and Martin J. Bastiaans discuss the application of linear canonical transformations to the modeling of light propagation through paraxial optical systems. Complementarily, M. Alper Kutay, Haldun M. Ozaktas, and José A. Rodrigo consider the use of simple optical systems for implementing linear canonical transformations, both in one and two variables in Chap. 6. The second part of the book focuses on practical aspects of the numerical implementation of linear canonical transformations. In Chap. 7, Figen S. Oktem and Haldun M. Ozaktas discuss the degrees of freedom involved in the implementation of a linear canonical transformation. The effects of sampling and discretization of linear canonical transformations are presented by John J.

Healy and Haldun M. Ozaktas in Chap. 8. Markus Testorf and Brian Hennelly investigate in Chap. 9 the effect known as self-imaging in systems described by linear canonical transformations. This part concludes with a discussion by Aykut Koç and Haldun M. Ozaktas in Chap. 10 about fast computational implementations of linear canonical transformations. The third and final part of the book is devoted to applications. This part opens with a study in Chap. 11 by Unnikrishnan Gopinathan, John Healy, Damien P. Kelly, and John T. Sheridan of the connection between linear canonical transformations and the retrieval of the phase of a field from the knowledge of its intensity. In Chap. 12, Damien P. Kelly and John T. Sheridan discuss the application of these transformations in digital holography. Applications to signal encryption are presented in Chap. 13 by Pramod Kumar, Joby Joseph and Kehar Singh. Steen G. Hanson, Michael L. Jakobsen and Harold T. Jura explore the use of these transformations for speckle metrology in Chap. 14. Lastly, the use of linear canonical transformations in quantum optics is presented by Gabriel F. Calvo and Antonio Picón.

This volume will be a very useful reference for specialists working in the fields of optical system design and modeling, image and signal processing, and quantum optics, to name a few. It will also be a great resource for graduate students in physics and engineering, as well as for scientists in other areas seeking to learn more about this important yet relatively unfamiliar class of integral transformations.

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Preface

Linear canonical transforms (LCTs) are a three-parameter family of linear integral transformations, which have a quadratic-phase kernel. For this reason, they have also been called quadratic-phase transforms or quadratic-phase systems (as well as other names). They are unitary transforms that correspond to linear, area-preserving distortions in phase space, a fact which underlies certain invariance properties. Combinations of LCTs are again LCTs. The family includes important operations or transforms such as chirp multiplication, chirp convolution (Fresnel transforms), fractional Fourier transforms, and of course the ordinary Fourier transform, as special cases. Arbitrary LCTs can be written as combinations of these simpler transforms. This leads to fast algorithms for approximately calculating LCTs, much as the ordinary Fourier transform can be calculated with fast algorithms.

LCTs have been rediscovered many times in different contexts, a fact we consider evidence of their ubiquity. Their significance in optics was recognized at least as early as the 1970s. Later, interest in the fractional Fourier transform during the 1990s led to renewed interest in LCTs from new perspectives.

This book deals with LCTs primarily from the perspective of signal and image processing, and optical information processing. Part I presents the mathematical theory of LCTs in the style of signal theory and analysis, as well as the foundations of how LCTs are related to optical systems. Part II deals with issues of degrees of freedom, sampling, numerical implementation, and fast algorithms. Part III is a survey of various applications. No attempt is made here to discuss canonical transformations as they appear in classical Hamiltonian mechanics and symplectomorphisms. These are well-established subjects in physics. However, we note that it is quite possible that a crossover of concepts and techniques between the different approaches to these transforms may be quite fruitful, and we hope this book may contribute to that end, in addition to being useful for its primary audience in the areas of signal processing and optics.

Overview

The opening chapters cover a range of fundamental topics. We start with a discussion of the twin discovery of LCTs in two different areas: paraxial optics and nuclear physics. This provides a fascinating window into more than 40 years of parallel scientific progress. This chapter also contrasts two parallel efforts to define a discrete counterpart to the LCTs—one based on group theory, the other on sampling theory. Chapter 2 provides a self-contained introduction to LCTs and their properties, so the reader who just wishes to dip into the subject may be advised to start here. Chapter 3 discusses the eigenfunctions of the LCTs. These functions are important for analyzing the characteristics of the transforms. Since the LCT can be used to describe wave propagation, they also play important roles in the analysis of self-imaging and resonance phenomena. Chapter 4 continues the theme of key properties of the transform with a discussion of the uncertainty principle. Heisenberg's principle provides a lower bound on the spread of signal energy in the time and frequency domains, and there has been a good deal of work on extending this work to LCTs. The first part of the book is rounded out by Chaps. 5 and 6 that discuss the relationship of LCTs to optics. These chapters deal with both how LCTs can be used to model and analyze optical systems and how LCTs can be optically implemented.

The modern age is digital, whether we are working with spatial light modulators and digital cameras or processing the resulting signals with a computer. In the second part of the book, we have a number of chapters on topics relevant to discrete signals and their processing. Chapter 7 discusses a modern interpretation of the relationship between sampling and information content of signals. Chapter 8 discusses sampling theory and builds up to a discrete transform. Periodic gratings have long been known to produce discrete signals at certain distances, and in Chap. 9 this Talbot effect and hence the relationship between discrete and periodic signals are examined. Just as the fast Fourier transform is key to the utility of conventional spectral analysis, corresponding fast algorithms are critical to our ability to use LCTs in a range of applications. Chapter 10 examines how to calculate the LCT numerically in a fast and accurate fashion.

In the final part of the book, we turn to a series of chapters in which linear canonical transforms are used in a variety of optical applications. One of the fundamental problems in optics is that our detectors are insensitive to phase. Chapter 11 discusses phase retrieval from the field intensity captured in planes separated by systems that can be described using LCTs, focusing particularly on non-iterative techniques. Another way to find the full wave field (amplitude and phase) is to record a hologram, a topic which experienced a revival in the past 20 years due to the rapid improvement in digital cameras. Digital holography is the focus of Chap. 12. Chapter 13 examines optical encryption by means of random phase encoding in multiple planes separated by systems that may be described using LCTs. Coherent light reflected from a rough surface develops laser speckle, a characteristic of the wave field, which may be beneficial in metrology or a nuisance in display technologies. Chapter 14 examines complex-parametered LCTs as a means of modelling speckle fields propagating through apertured optical systems. With Chap. 15, the book is rounded off with a discussion of the use of LCTs in quantum optics.

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Contents

Part I
Fundamentals

Chapter 1
Development of Linear Canonical Transforms: A Historical Sketch

Kurt Bernardo Wolf

Abstract Linear canonical transformations (LCTs) were introduced almost simultaneously during the early 1970s by Stuart A. Collins Jr. in paraxial optics, and independently by Marcos Moshinsky and Christiane Quesne in quantum mechanics, to understand the conservation of information and of uncertainty under linear maps of phase space. Only in the 1990s did both sources begin to be referred jointly in the growing literature, which has expanded into a field common to applied optics, mathematical physics, and analogic and digital signal analysis. In this introductory chapter we recapitulate the construction of the LCT integral transforms, detailing their Lie-algebraic relation with second-order differential operators, which is the origin of the metaplectic phase. Radial and hyperbolic LCTs are reviewed as unitary integral representations of the two-dimensional symplectic group, with complex extension to a semigroup for systems with loss or gain. Some of the more recent developments on discrete and finite analogues of LCTs are commented with their concomitant problems, whose solutions and alternatives are contained the body of this book.

1.1 Introduction

The discovery and development of the theory of linear canonical transforms (LCTs) during the early seventies was motivated by the work on two rather different physical models: paraxial optics and nuclear physics. The integral LCT kernel was written as a descriptor for light propagation in the paraxial régime by Stuart A. Collins Jr., working in the ElectroScience Laboratory of Electrical Engineering at Ohio State University. On the other hand, Marcos Moshinsky and his postdoctoral associate Christiane Quesne, theoretical physicists at the Institute of Physics of the Universidad Nacional Autónoma de México, while working among other problems on the alpha clustering and decay of radioactive nuclei, saw LCTs as

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