Classical and Quantum Dynamics

from Classical Paths to Path Integrals

Second Corrected
and Enlarged Edition

经典和量子动力学

第2版

W. Dittrich M. Reuter

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Professor Dr. Walter Dittrich

Institut für Theoretische Physik, Universität Tübingen,
Auf der Morgenstelle 14, D-72076 Tübingen, Germany

Dr. Martin Reuter

Institut für Theoretische Physik, Universität Hannover,
Appelstrasse 2, D-30167 Hannover, Germany

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Preface to the Second Edition

In this second enlarged edition we have supplemented the chapters on geometric phases. We have also added a new chapter on anyon physics in planar electrodynamics. Finally we have corrected some minor typographical errors. One of us (W.D.) wants to thank the "Volkswagen-Stiftung" for its generous financial support during his Sabbatical in the U.S., where the present improved version took shape.

Tübingen and Hamburg
October 1993

Walter Dittrich
Martin Reuter

Preface to the First Edition

This volume is the result of the authors' lectures and seminars given at Tübingen University and elsewhere. It represents a summary of our learning process in nonlinear Hamiltonian dynamics and path integral methods in nonrelativistic quantum mechanics. While large parts of the book are based on standard material, readers will find numerous worked examples which can rarely be found in the published literature. In fact, toward the end they will find themselves in the midst of modern topological methods which so far have not made their way into the textbook literature.

One of the authors' (W.D.) interest in the subject was inspired by Prof. D. Judd (UC Berkeley), whose lectures on nonlinear dynamics familiarized him with Lichtenberg and Lieberman's monograph, *Regular and Stochastic Motion* (Springer, 1983). For people working in plasma or accelerator physics, the chapter on nonlinear physics should contain some familiar material. Another influential author has been Prof. J. Schwinger (UCLA); the knowledgeable reader will not be surprised to discover our appreciation of Schwinger's Action Principle in the introductory chapters. However, the major portion of the book is based on Feynman's path integral approach, which seems to be the proper language for handling topological aspects in quantum physics.

Our thanks go to Ginny Dittrich for masterly transforming a long and complex manuscript into a readable monograph.

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Walter Dittrich
Martin Reuter

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Contents

Introduction

The subject of this monograph is classical and quantum dynamics. We are fully aware that this combination is somewhat unusual, for history has taught us convincingly that these two subjects are founded on totally different concepts; a smooth transition between them has so far never been made and probably never will.

An approach to quantum mechanics in purely classical terms is doomed to failure; this fact was well known to the founders of quantum mechanics. Nevertheless, to this very day people are still trying to rescue as much as possible of the description of classical systems when depicting the atomic world. However, the currently accepted viewpoint is that in describing fundamental properties in quantum mechanics, we are merely borrowing names from classical physics. In writing this book we have made no attempt to contradict this point of view. But in the light of modern topological methods we have tried to bring a little twist to the standard approach that treats classical and quantum physics as disjoint subjects.

The formulation of both classical and quantum mechanics can be based on the principle of stationary action. Schwinger has advanced this principle into a powerful working scheme which encompasses almost every situation in the classical and quantum worlds. Our treatment will give a modest impression of the wide range of applicability of Schwinger's action principle.

We then proceed to rediscover the importance of such familiar subjects as Jacobi fields, action angle variables, adiabatic invariants, etc. in the light of current research on classical Hamiltonian dynamics. It is here that we recognize the important role that canonical perturbation theory played before the advent of modern quantum mechanics.

Meanwhile, classical mechanics has been given fresh impetus through new developments in perturbation theory, offering a new look at old problems in nonlinear mechanics like, e.g., the stability of the solar system. Here the KAM theorem proved that weakly disturbed integrable systems will remain on invariant surfaces (tori) for most initial conditions and do not leave the tori to end up in chaotic motion.

At this stage we point to the fundamental role that adiabatic invariants played prior to canonical quantization of complementary dynamical variables. We are reminded of torus quantization, which assigns each adiabatic invariant an integer multiple of Planck's constant. All these semiclassical quantization procedures have

much in common with Feynman's path integral or, rather, approximations thereof. Indeed, Feynman's path integral methods are ideally suited to follow a quantum mechanical system – if certain restrictions are enforced – into its classical realm. Consequently it is one of our main goals to apply Feynman's path integral and other geometrical methods to uncover the mystery of the zero point energy (Maslov anomaly) of the quantum harmonic oscillator.

That quantum and classical mechanics are, in fact, disjoint physical worlds was clear from the very beginning. Present-day experience is no exception; it is rather embarrassing to find out that an important geometric phase in a cyclic adiabatic quantal process has been overlooked since the dawn of quantum mechanics. This so-called Berry phase signals that in nonrelativistic as well as relativistic quantum theory, geometrical methods play an eminent role.

The appearance of topology in quantum mechanics is probably the most important new development to occur in recent years. A large portion of this text is therefore devoted to the geometric structure of topologically nontrivial physical systems. Berry phases, Maslov indices, Chern-Simons terms and various other topological quantities have clearly demonstrated that quantum mechanics is not, as of yet, a closed book.

1. The Action Principles in Mechanics

We begin this chapter with the definition of the action functional as time integral over the Lagrangian $L(q_i(t), \dot{q}_i(t); t)$ of a dynamical system:

$$S\{[q_i(t)]; t_1, t_2\} = \int_{t_1}^{t_2} dt\, L(q_i(t), \dot{q}_i(t); t)\,. \tag{1.1}$$

Here, q_i, $i = 1, 2, \ldots, N$, are points in N-dimensional configuration space. Thus $q_i(t)$ describes the motion of the system, and $\dot{q}_i(t) = dq_i/dt$ determines its velocity along the path in configuration space. The endpoints of the trajectory are given by $q_i(t_1) = q_{i1}$, and $q_i(t_2) = q_{i2}$.

Next we want to find out what the actual dynamical path of the system is. The answer is contained in the principle of stationary action: in response to infinitesimal variation of the integration path, the action S is stationary, $\delta S = 0$, for variations about the correct path, provided the initial and final configurations are held fixed. On the other hand, if we permit infinitesimal changes of $q_i(t)$ at the initial and final times, including alterations of those times, the only contribution to δS comes from the endpoint variations, or

$$\delta S = G(t_2) - G(t_1)\,. \tag{1.2}$$

Equation (1.2) is the most general formulation of the action principle in mechanics. The fixed values G_1 and G_2 depend only on the endpoint path variables at the respective terminal times.

Again, given a system with the action functional S, the actual time evolution in configuration space follows that path about which general variations produce only endpoint contributions. The explicit form of G is dependent upon the special representation of the action principle. In the following we begin with the one that is best known, i.e.,

1) Lagrange: The Lagrangian for a point particle with mass m, moving in a potential $V(x_i, t)$, is

$$L(x_i, \dot{x}_i; t) = \frac{m}{2}\dot{x}_i^2 - V(x_i, t)\,. \tag{1.3}$$

Here and in the following we restrict ourselves to the case $N = 3$; i.e., we describe the motion of a single mass point by $x_i(t)$ in real space. The dynamical variable

$x_i(t)$ denotes the actual classical trajectory of the particle which is parametrized by t with $t_1 \leq t \leq t_2$.

Now we consider the response of the action functional (1.1) with respect to changes in the coordinates and in the time, $\delta x_i(t)$ and $\delta t(t)$, respectively. It is important to recognize that, while the original trajectory is being shifted in real space according to

$$x_i(t) \to x_i'(t') = x_i(t) + \delta x_i(t) \tag{1.4}$$

the time-readings along the path become altered locally, i.e., different at each individual point on the varied curve – including the endpoints. This means that our time change is *not* a global ($\delta t(t) = \text{const.}$) rigid time displacement, equally valid for all points on the trajectory, but that the time becomes changed locally, or, shall we say, gauged, for the transported trajectory. All this indicates that we have to supplement (1.4) by

$$t \to t'(t) = t + \delta t(t)\,, \tag{1.5}$$

where the terminal time changes are given by $\delta t(t_2) = \delta t_2$, and $\delta t(t_1) = \delta t_1$.

To the time change (1.5) is associated the change in the integration measure in (1.1) given by the Jacobi formula

$$d(t + \delta t) = \frac{d(t + \delta t)}{dt}\, dt = \left(1 + \frac{d}{dt}\, \delta t(t)\right) dt \tag{1.6}$$

or

$$\delta(dt) := d(t + \delta t) - dt = dt \frac{d}{dt}\, \delta t(t)\,. \tag{1.7}$$

If the time is not varied, we write δ_0 instead of δ; i.e., $\delta_0 t = 0$ or $[\delta_0, d/dt] = 0$. The variation of $x_i(t)$ is then given by

$$\delta x_i(t) = \delta_0 x_i(t) + \delta t \frac{d}{dt}(x_i(t)) \tag{1.8}$$

since up to higher order terms we have

$$\begin{aligned}\delta x_i(t) &= x_i'(t') - x_i(t) = x_i'(t + \delta t) - x_i(t) = x_i'(t) + \delta t \frac{dx_i'(t)}{dt} - x_i(t) \\ &= (x_i'(t) - x_i(t)) + \delta t \frac{dx_i}{dt} =: \delta_0 x_i(t) + \delta t \frac{dx_i}{dt}\,.\end{aligned}$$

Similarly,

$$\delta \dot{x}_i(t) = \delta_0 \dot{x}_i(t) + \delta t \frac{d}{dt}\, \dot{x}_i \tag{1.9}$$

$$= \delta_0 \dot{x}_i + \frac{d}{dt}(\delta t \dot{x}_i) - \dot{x}_i \frac{d}{dt}\,(\delta t)$$

$$= \frac{d}{dt}\left(\delta_0 + \delta t \frac{d}{dt}\right) x_i - \dot{x}_i \frac{d}{dt} \delta t = \frac{d}{dt}(\delta x_i) - \dot{x}_i \frac{d}{dt}\, \delta t\,. \tag{1.10}$$

The difference between δ and δ_0 acting on t, $x_i(t)$ and $\dot{x}_i(t)$ is expressed by the identity

$$\delta = \delta_0 + \delta t \frac{d}{dt} . \tag{1.11}$$

So far we have obtained

$$\delta S = \int_{t_1}^{t_2} [\delta(dt)L + dt\delta L] = \int_{t_1}^{t_2} dt \left[\overbrace{L\frac{d}{dt}(\delta t)}^{\frac{d}{dt}(L\delta t) - \delta t \frac{dL}{dt}} + \delta L \right]$$

$$= \int_{t_1}^{t_2} dt \left[\frac{d}{dt}(L\delta t) + \left(\delta L - \delta t \frac{dL}{dt} \right) \right] = \int_{t_1}^{t_2} dt \left[\frac{d}{dt}(L\delta t) + \delta_0 L \right] , \tag{1.12}$$

since, according to (1.11) we have

$$\delta L = \delta_0 L + \delta t \frac{d}{dt} L . \tag{1.13}$$

The total variation of the Lagrangian is then given by

$$\begin{aligned} \delta L = \delta_0 L + \delta t \frac{d}{dt} L &= \frac{\partial L}{\partial x_i} \delta_0 x_i + \frac{\partial L}{\partial \dot{x}_i} \delta_0 \dot{x}_i + \delta t \frac{dL}{dt} \\ &= \frac{\partial L}{\partial x_i} \delta_0 x_i + \frac{\partial L}{\partial \dot{x}_i} \delta_0 \dot{x}_i + \delta t \left(\frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i + \frac{\partial L}{\partial t} \right) \\ &= \frac{\partial L}{\partial x_i} \left(\delta_0 + \delta t \frac{d}{dt} \right) x_i + \frac{\partial L}{\partial \dot{x}_i} \left(\delta_0 + \delta t \frac{d}{dt} \right) \dot{x}_i + \delta t \frac{\partial L}{\partial t} . \\ &= \frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i + \frac{\partial L}{\partial t} \delta t . \end{aligned}$$

Now we go back to (1.3) and substitute

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V(x_i, t)}{\partial x_i} , \quad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i , \quad \frac{\partial L}{\partial t} = -\frac{\partial V}{\partial t} , \tag{1.14}$$

so that we obtain, with the aid of (1.10):

$$\delta L = -\frac{\partial V}{\partial t} \delta t - \frac{\partial V}{\partial x_i} \delta x_i + m\dot{x}_i \frac{d}{dt} \delta x_i - m\dot{x}_i^2 \frac{d}{dt} \delta t . \tag{1.15}$$

Our expression for δS then becomes

$$\delta S = \int_{t_1}^{t_2} dt \left[m\dot{x}_i \frac{d}{dt} \delta x_i - \frac{\partial V}{\partial t} \delta t - \frac{\partial V}{\partial x_i} \delta x_i + (L(t) - m\dot{x}_i^2) \frac{d}{dt} \delta t \right] . \tag{1.16}$$

We can also write the last expression for δS a bit differently, thereby presenting explicitly the coefficients of δx_i and δt:

$$\delta S = \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt}\left[m\frac{dx_i}{dt}\,\delta x_i - \left(\frac{m}{2}\left(\frac{dx_i}{dt}\right)^2 + V\right)\delta t\right] \right.$$
$$\left. -m\frac{d^2x_i}{dt^2}\,\delta x_i - \frac{\partial V}{\partial x_i}\,\delta x_i - \frac{\partial V}{\partial t}\,\delta t + \delta t\frac{d}{dt}\left[\frac{m}{2}\left(\frac{dx_i}{dt}\right)^2 + V\right]\right\}, \quad (1.17)$$

or with the definition

$$E = \frac{\partial L}{\partial \dot{x}_i}\,\dot{x}_i - L = \frac{m}{2}\left(\frac{dx_i}{dt}\right)^2 + V(x_i, t)\,, \quad (1.18)$$

$$\delta S = \int_{t_1}^{t_2} dt \frac{d}{dt}\left[m\frac{dx_i}{dt}\delta x_i - E\delta t\right]$$
$$+ \int_{t_1}^{t_2} dt \left[-\delta x_i\left(m\frac{d^2x_i}{dt^2} + \frac{\partial V}{\partial x_i}\right) + \delta t\left(\frac{dE}{dt} - \frac{\partial V}{\partial t}\right)\right]. \quad (1.19)$$

Since δx_i and δt are independent variations, the action principle $\delta S = G_2 - G_1$ implies the following laws:

$$\delta x_i:\ m\frac{d^2x_i}{dt^2} = -\frac{\partial V(x_i, t)}{\partial x_i}\,, \quad \text{(Newton)}\,, \quad (1.20)$$

i.e., *one* second-order differential equation.

$$\delta t:\ \frac{dE}{dt} = \frac{\partial V}{\partial t}\,, \quad (1.21)$$

so that for a static potential, $\partial V/\partial t = 0$, the law of the conservation of energy follows: $dE/dt = 0$.

$$\text{Surface term:}\quad G = m\frac{dx_i}{dt}\,\delta x_i - E\delta t\,. \quad (1.22)$$

2) Hamiltonian: As a function of the Hamiltonian,

$$H(x_i, p_i; t) = \frac{p_i^2}{2m} + V(x_i, t)\,, \quad (1.23)$$

the Lagrangian (1.3) can also be written as ($p_i := \partial L/\partial \dot{x}_i$):

$$L = p_i\frac{dx_i}{dt} - H(x_i, p_i; t)\,. \quad (1.24)$$

Here, the independent dynamical variables are x_i and p_i; t is the independent time-parameter variable. Hence the change of the action is

$$\delta S = \delta\int_{t_1}^{t_2} dt \left[p_i\frac{dx_i}{dt} - H(x_i, p_i; t)\right]$$
$$= \int_{t_1}^{t_2} dt \left[p_i\frac{d}{dt}\,\delta x_i + \frac{dx_i}{dt}\,\delta p_i - \delta H - H\frac{d}{dt}\,\delta t\right]. \quad (1.25)$$

Upon using

$$\delta H = \left(\frac{\partial H}{\partial x_i}\,\delta x_i + \frac{\partial H}{\partial p_i}\,\delta p_i\right) + \frac{\partial H}{\partial t}\,\delta t\ , \tag{1.26}$$

where, according to (1.23): $\partial H/\partial x_i = \partial V/\partial x_i$ and $\partial H/\partial p_i = p_i/m$, we obtain

$$\delta S = \int_{t_1}^{t_2} dt \frac{d}{dt}[p_i \delta x_i - H\delta t]$$
$$+ \int_{t_1}^{t_2} dt \left[-\delta x_i \left(\frac{dp_i}{dt} + \frac{\partial V}{\partial x_i}\right) + \delta p_i \left(\frac{dx_i}{dt} - \frac{p_i}{m}\right) + \delta t \left(\frac{dH}{dt} - \frac{\partial H}{\partial t}\right)\right]\ . \tag{1.27}$$

The action principle $\delta S = G_2 - G_1$ then tells us here that

$$\delta p_i : \ \frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m}\ , \tag{1.28}$$

$$\delta x_i : \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} = -\frac{\partial V}{\partial x_i}\ . \tag{1.29}$$

Here we recognize the two first-order Hamiltonian differential equations.

$$\delta t : \ \frac{dH}{dt} = \frac{\partial H}{\partial t}\ . \tag{1.30}$$

$$\text{Surface term} : \quad G = p_i \delta x_i - H\delta t\ . \tag{1.31}$$

Let us note for later use:

$$\delta S = G_2 - G_1 = [p_i \delta x_i - H\delta t]_{t_2} - [p_i \delta x_i - H\delta t]_{t_1}\ . \tag{1.32}$$

Compared with $(x_1 := \{x_i(t_1)\},\ x_2 := \{x_i(t_2)\};\ i = 1,2,3)$

$$\delta S = \frac{\partial S}{\partial x_1}\,\delta x_1 + \frac{\partial S}{\partial x_2}\,\delta x_2 + \frac{\partial S}{\partial t_1}\,\delta t_1 + \frac{\partial S}{\partial t_2}\,\delta t_2 \tag{1.33}$$

(1.32) yields

$$p_1 = -\frac{\partial S}{\partial x_1}\ , \quad H(x_1, p_1; t_1) = \frac{\partial S}{\partial t_1} \tag{1.34}$$

or

$$H\left(x_1, -\frac{\partial S}{\partial x_1}, t_1\right) - \frac{\partial S}{\partial t_1} = 0\ . \tag{1.35}$$

In the same manner, it follows that:

$$p_2 = \frac{\partial S}{\partial x_2}\ , \quad H\left(x_2, \frac{\partial S}{\partial x_2}, t_2\right) + \frac{\partial S}{\partial t_2} = 0\ . \tag{1.36}$$

Obviously, (1.35) and (1.36) are the Hamilton-Jacobi equations for finding the action S. In this way we have demonstrated that the action (1.1) satisfies the Hamilton-Jacobi equation. (Later on we shall encounter S again as the generating function of a canonical transformation $(q_i, p_i) \to (Q_i, P_i)$ of the $F_1(q_i, Q_i, t)$-type.

3) Euler-Maupertuis (Principle of Least Action): This principle follows from the Lagrangian representation of the action principle:

$$\delta S = \delta \int_{t_1}^{t_2} dt\, L = \left[m \frac{dx_i}{dt}\, \delta x_i - E \delta t \right]_1^2 , \tag{1.37}$$

if we introduce the following restrictions:

a) L should not be explicitly time dependent; then the energy E is a conserved quantity both on the actual and the varied paths; b) for the varied paths, $\delta x_i(t)$ should vanish at the terminal points: $\delta x_i(t_{1,2}) = 0$. What remains is

$$\delta \int_{t_1}^{t_2} dt\, L = -E(\delta t_2 - \delta t_1) . \tag{1.38}$$

But under the same restrictions we have, using (1.18),

$$\int_{t_1}^{t_2} dt\, L = \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{x}_i}\, \dot{x}_i - E(t_2 - t_1) , \tag{1.39}$$

the variation of which is given by

$$\delta \int_{t_1}^{t_2} dt\, L = \delta \int_{t_1}^{t_2} dt \frac{\partial L}{\partial \dot{x}_i}\, \dot{x}_i - E(\delta t_2 - \delta t_1) . \tag{1.40}$$

Comparing (1.40) with (1.38), we get, taking into consideration $p_i := \partial L / \partial \dot{x}_i$:

$$\delta \int_{t_1}^{t_2} dt\, p_i \frac{dx_i}{dt} = 0 . \tag{1.41}$$

If, in addition, we assume the potential to be independent of the velocity, i.e., that

$$\frac{\partial T}{\partial \dot{x}_i}\, \dot{x}_i = 2T , \tag{1.42}$$

then (1.41) takes on the form

$$\delta \int_{t_1}^{t_2} dt\, T = 0 , \tag{1.43}$$

or

$$\int_{t_1}^{t_2} dt\, T = \text{Extremum.} \tag{1.44}$$

Thus the Euler-Maupertuis Principle of Least Action states: The time integral of the kinetic energy of the particle is an extreme value for the path actually selected compared to the neighboring paths with the same total energy which the particle will travel between the initial and final position at any time – t is varied! This variation in time can also be expressed by writing (1.43) in the form [see also (1.7)]:

$$\delta \int_{t_1}^{t_2} dt\, T = \int_{t_1}^{t_2} dt \left(T \frac{d}{dt} \delta t + \delta T \right). \tag{1.45}$$

In N-dimensional configuration space, (1.41) is written as

$$\delta \int_{t_1}^{t_2} \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \, dt = 0 \,, \tag{1.46}$$

or

$$\delta \int_{1}^{2} \sum_{i=1}^{N} p_i \, dq_i = 0 \,. \tag{1.47}$$

If we parametrize the path in configuration space between 1 and 2 using the parameter ϑ, then (1.47) is written

$$\delta \int_{\vartheta_1}^{\vartheta_2} \sum_{i=1}^{N} p_i \frac{dq_i}{d\vartheta} \, d\vartheta = 0 \,. \tag{1.48}$$

On the other hand, it follows from the Hamiltonian version of the action principle in its usual form with vanishing endpoint contributions $\delta q_i(t_{1,2}) = 0$, $\delta t(t_{1,2}) = 0$ in $2N$-dimensional phase space:

$$\tilde{\delta} \int_{t_1}^{t_2} dt \left[\sum_{i=1}^{N} p_i \frac{dq_i}{dt} - H \right] = 0 \tag{1.49}$$

One should note the different role of δ in (1.46) – the time is also varied – and $\tilde{\delta}$, which stands for the conventional virtual (timeless) displacement.

With the parametrization ϑ in (1.49), the expression

$$\tilde{\delta} \int_{\vartheta_1}^{\vartheta_2} d\vartheta \left[\sum_{i=1}^{N} p_i \frac{dq_i}{d\vartheta} - H \frac{dt}{d\vartheta} \right] = 0 \tag{1.50}$$

can, by introducing conjugate quantities,

$$q_{N+1} = t \,, \quad p_{N+1} = -H \,, \tag{1.51}$$

be reduced formally to a form similar to (1.48):

$$\tilde{\delta} \int_{\vartheta_1}^{\vartheta_2} \sum_{i=1}^{N+1} p_i \frac{dq_i}{d\vartheta} \, d\vartheta = 0 \,. \tag{1.52}$$

Besides the fact that in (1.52) we have another pair of canonical variables, the different roles of the two variation symbols δ and $\tilde{\delta}$ should be stressed. δ refers to the paths with constant $H = E$, whereas in the $\tilde{\delta}$ variation, H can, in principle, be any function of time. $\tilde{\delta}$ in (1.52) applies to $2N + 2$-dimensional phase space, while δ in (1.48) applies to configuration space.

If, in the case of the principle of least action, no external forces are involved, i.e., we set without loss of generality $V = 0$, then E as well as T are constants. Consequently, the Euler-Maupertuis principle takes the form

$$\delta \int_{t_1}^{t_2} dt = 0 = \delta t_2 - \delta t_1 \,, \tag{1.53}$$

i.e., the time along the actual dynamical path is an extremum.

At this point we are reminded of Fermat's principle of geometrical optics: A light ray selects that path between two points which takes the shortest time to travel.

Jacobi proposed another version of the principle of least action. It is always useful when one wishes to construct path equations in which time does not appear. We derive this principle by beginning with the expression for the kinetic energy of a free particle in space:

$$T = \frac{1}{2} \sum_{i,k=1}^{3} m_{ik} \frac{dx_i}{dt} \frac{dx_k}{dt} \,, \tag{1.54}$$

where m_{ik} are the elements of the mass tensor, e.g. $m_{ik} = m\delta_{ik}$.

In generalized coordinates in N-dimensional configuration space, we then have

$$T = \frac{1}{2} \frac{(ds)^2}{(dt)^2} \,, \tag{1.55}$$

with the line element

$$(ds)^2 = \sum_{i,k=1}^{N} m_{ik}(q_1, q_2, \ldots, q_N) dq_i dq_k \tag{1.56}$$

and position-dependent elements m_{ik}; for example, from

$$T = \frac{m}{2} \frac{(dr)^2 + r^2(d\vartheta)^2 + (dz)^2}{(dt)^2} \tag{1.57}$$

we can immediately see that

$$\overleftrightarrow{m} = \begin{pmatrix} m & 0 & 0 \\ 0 & mr^2 & 0 \\ 0 & 0 & m \end{pmatrix} .$$

The m_{ik} take over the role of the metric tensor in configuration space. At this point mechanics becomes geometry.

Writing (1.55) in the form $dt = ds/\sqrt{2T}$ we can restate (1.43) as

$$\delta \int_{t_1}^{t_2} dt\, T = 0 = \delta \int_1^2 ds \sqrt{T} \,. \tag{1.58}$$

Here, we substitute $T = H - V(q_i)$ to obtain Jacobi's principle: