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# The Effect of Slow Flow Dynamics on the Oscillations of a Singular Damped System with an Essentially Nonlinear Attachment

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**Abstract**

We study a three degree of freedom autonomous system with damping, composed of two linear coupled oscillators with an essentially nonlinear lightweight attachment. In particular, we are interested in strongly nonlinear interactions between the linear oscillators and the essentially nonlinear attachment. First, we reduce our system to a non-autonomous second order nonlinear damped oscillator. Then, we introduce a slow-fast partition of the dynamics and average out the main frequency components in order to obtain a reduced system that is studied through the Slow Invariant Manifold (SIM) approach. Depending on the parameters of the system we find different interesting nonlinear dynamical phenomena. With the help of the SIM approach we can study how the parameters of the original problem influence the asymptotic behavior of the orbits of the system. This is accomplished with the application of Tikhonov's theorem. We classify the different cases of the dynamics according to the values of the parameters and the theoretically predicted asymptotic behavior of the orbits. Interesting phenomena are reported such as orbit capture, relaxation oscillations and complex structure of the basins of attraction.



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## 1 Introduction

In mechanical applications, there is a great interest in structures of linear systems with nonlinear attachments [1–5]. In most of these studies the nonlinear attachments have small masses in comparison to the structures to which they are attached and the systems are non conservative. Various dynamical phenomena have been investigated for two degrees of freedom damped systems with or without external forcing. A special property of these configurations is that the nonlinear substructures can act as nonlinear energy sinks (NESs) and absorb, through irreversible transient transfer, energy from the linear parts. There has been shown, both numerically and analytically, that the above can occur through resonance captures in vicinities of resonance manifolds of the underlying conservative systems [2, 5] for certain ranges of parameters and initial conditions. In the case of external forcing various other complicated dynamical phenomena may appear [1,5,6].

These systems can also be studied with the help of the method of multiple scales and singularity analysis [1, 5–9]. It has been shown that the Slow Invariant Manifolds (SIMs) obtained in the singular limit plays a very important role in the dynamics under consideration. If a SIM is stable it can attract the orbits of the system in the limit of 'fast' time scales for a wide range of initial conditions and accounts for transient energy transfer. If it is unstable, for example of saddle type, it has locally invariant manifolds and their existence affect the dynamics of such systems and may produce relaxation oscillations and complex phenomena [7,8,10,11].

We study a three degree of freedom autonomous system considered previously in [3,4]. Through singular transformations and reduction of the dynamics, we derive a reduced non autonomous damped strongly nonlinear second order differential equation. With the use of the complexification-averaging technique (CX-A) we obtain a system of two, first order, differential equations governed by the slow time scale. The Slow Invariant Manifold (SIM) of the above system, computed through multiple scale analysis or singularity analysis [9], provides information about the asymptotic behavior of the orbits of the reduced system. The SIM of the system may have one or three branches depending on the slow time and the value of the damping parameter. The corresponding bifurcations affect the dynamical behavior of the system which can change drastically. Indeed, Tikhonov's theorem guarantees that, when the branches of the SIM are isolated and stable, the orbits of the system tend to these stable branches as long as they exist.

The application of Tikhonov's theorem allows us to classify the behavior of the orbits in comparison to the evolution of the SIM, depending on the system parameters. The theoretical analysis provides the range of the system parameters for each of the previous cases and allows us to predict the long term behavior of the orbits of the reduced system. In fact, the damping parameter $\lambda$ and the number of the branches of the SIM play an essential role in the evolution of the dynamics of the system. The dynamics can be simple, as in the case where there is only one persisting stable branch of the SIM, or complicated, when bifurcations of the SIM occur, resulting in different phenomena such as relaxation oscillations, orbit excitations and complex structures of the basins of attraction, especially when the damping parameter is small.

The manuscript is organized as follows. In the next section, we present the system and its reduction through singular transformations and the complexification averaging technique(CX-A). In the third section, we apply multiple scale analysis to the resulting system and compute the SIM. The polynomial, that defines it, depends on the slow time. We study the change of the number of its roots, depending on time and system parameters, and determine the bifurcations and stability of the different branches of the SIM. Based on Tichonov's theorem we find the range of parameters and the interval of time where the branches of the SIM are attracting. In the fourth section, we present numerical results and classify the behavior of the orbits. Finally we present some concluding remarks.

## 2 Reduction of the system

The initial system considered in this work is composed of two coupled linear oscillators and a nonlinear oscillator that interacts through an essential nonlinearity with one of the linear oscillators. The mass of the nonlinear attachment is small in comparison to the masses of the linear oscillators and therefore the problem is singular. The governing equations of motion are given by,

$$\begin{aligned} &\varepsilon\ddot{y}+\varepsilon\lambda(\dot{y}-\dot{x}_0)+C(y-x_0)^3=0,\\ &\ddot{x}_0+d(x_0-x_1)=\varepsilon\lambda(\dot{y}-\dot{x}_0)+C(y-x_0)^3,\\ &\ddot{x}_1+ax_1+d(x_1-x_0)=0, \end{aligned} \tag{1}$$

where $a,d,C,\lambda$, and $\varepsilon << 1$ are the parameters of the system. After applying the linear singular transformation $v=\varepsilon^{-1/2}x_0+\varepsilon^{1/2}y$, $w=\varepsilon^{-1/2}x_0-\varepsilon^{-1/2}y$ the system assumes the form:

$$\begin{aligned} &\ddot{w}+\lambda(1+\varepsilon)\dot{w}+C(1+\varepsilon)w^3=-d\frac{v+\varepsilon w}{1+\varepsilon}+dx_1,\\ &\ddot{v}+\frac{d}{1+\varepsilon}v-dx_1=-\frac{\varepsilon dw}{1+\varepsilon},\\ &\ddot{x}_1+(a+d)x_1-\frac{d}{1+\varepsilon}v=\frac{\varepsilon dw}{1+\varepsilon}. \end{aligned} \tag{2}$$

The second and third equations of (2) form a non homogeneous set of linear ordinary differential equations with constant coefficients, eigenfrequencies of the homogeneous part

$$\begin{aligned} \omega_1^2&=\frac{(1+\varepsilon)a+(2+\varepsilon)d-\sqrt{-4ad(1+\varepsilon)+((1+\varepsilon)a+(2+\varepsilon)d)^2}}{2(1+\varepsilon)},\\ \omega_2^2&=\frac{(1+\varepsilon)a+(2+\varepsilon)d+\sqrt{-4ad(1+\varepsilon)+((1+\varepsilon)a+(2+\varepsilon)d)^2}}{2(1+\varepsilon)} \end{aligned} \tag{3}$$

and corresponding eigenvectors, of the homogeneous part, $(K_1,1)$ and $(K_2,1)$, where

$$\begin{aligned} K_1&=\frac{(1+\varepsilon)a+\varepsilon d-\sqrt{-4ad(1+\varepsilon)+((1+\varepsilon)a+(2+\varepsilon)d)^2}}{2d},\\ K_2&=\frac{(1+\varepsilon)a+\varepsilon d+\sqrt{-4ad(1+\varepsilon)+((1+\varepsilon)a+(2+\varepsilon)d)^2}}{2d}. \end{aligned} \tag{4}$$

We introduce modal coordinates for the linear part of Eqs.(2) through the coordinate transformations, $v=K_1z_1+K_2z_2$, $x_1=z_1+z_2$, and assume zero initial displacements and nonzero initial velocities $\dot{z}_1(0)=\omega_1z_{10}$, $\dot{z}_2(0)=\omega_2z_{20}$. Then we obtain the following approximate reduced system

$$\ddot{w}+\lambda\dot{w}+Cw^3=A\sin\omega_1t+B\sin\omega_2t+O(\varepsilon), \tag{5}$$

where we have replaced the solutions of the non homogeneous linear oscillators and where

$$
\begin{aligned}
A &= (d - \frac{dK_1}{1+\varepsilon})z_{10}, \\
B &= (d - \frac{dK_2}{1+\varepsilon})z_{20}.
\end{aligned} \tag{6}
$$

After performing a time transformation $\hat{t} \to \omega_{20}t$, where $\omega_2 = \omega_{20} + \varepsilon\bar{B}$, the above equation becomes

$$
w'' + \hat{\lambda}w' + \hat{C}w^3 = \hat{A}\sin\,(\frac{\omega_1}{\omega_{20}}\hat{t}) + \hat{B}\sin\,(1 + \frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}) + O(\varepsilon), \tag{7}
$$

where

$$
\hat{C} = \frac{C}{\omega_{20}^2}, \hat{\lambda} = \frac{\lambda}{\omega_{20}}, \hat{A} = \frac{A}{\omega_{20}^2}, \hat{B} = \frac{B}{\omega_{20}^2}. \tag{8}
$$

Considering the reduced system (7) we apply the complexification-averaging technique [5, 12] by introducing the complex coefficient $\Psi = w' + jw$ , so that equation (7) yields

$$
\Psi' + (\frac{\hat{\lambda}}{2} - \frac{j}{2})(\Psi + \Psi^*) + \frac{\hat{C}j}{8}(\Psi - \Psi^*)^3 = \hat{A}\sin(\frac{\omega_1}{\omega_{20}}\hat{t}) + \hat{B}\sin((1 + \frac{\varepsilon\bar{B}}{\omega_{20}})\hat{t}). \tag{9}
$$

At this point we make the important additional assumption that the considered dynamics is dominated by a single normalized 'fast' frequency equal to unity. It follows that our results will only be valid as long as this assumption is satisfied. Mathematically, this assumption is imposed by expressing $\Psi = \varphi e^{jt}$ so that Eq. (9) becomes

$$
\begin{aligned}
&\phi' + j\phi + (\frac{\hat{\lambda}}{2} - \frac{j}{2})(\phi + \phi^* e^{-2j\hat{t}}) + \frac{\hat{C}j}{8}(\phi^3 e^{2j\hat{t}} - 3\phi^2\phi^* + 3\phi(\phi^*)^2 e^{-2j\hat{t}} - (\phi^*)^3 e^{-4j\hat{t}}) \\
&= -\frac{j\hat{A}}{2}(e^{j\frac{\omega_1 - \omega_{20}}{\omega_{20}}\hat{t}} - e^{j\frac{-\omega_1 - \omega_{20}}{\omega_{20}}\hat{t}}) - \frac{\hat{B}j}{2}(e^{j\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}} - e^{-j(2 + \frac{\varepsilon\bar{B}}{\omega_{20}})\hat{t}}).
\end{aligned} \tag{10}
$$

We mention at this point that system (10) is approximate since it takes into account only a single 'fast' frequency. Clearly, since the system under consideration (and the reduced system (7)) is strongly nonlinear, this is only an approximation since higher harmonics will exist in the dynamics. However, we conjecture that there exist regimes where the harmonic components with normalized frequency unity dominate, and these will be analyzed in what follows. If we average out the harmonic terms with fast frequencies equal to unity or multiples of it, by integrating (10) over the common period $2\pi$ of the integral periods, and leaving out $O(\varepsilon)$ terms of the above equation, we derive the following averaged complex dynamical system

$$
\phi' + (\frac{\hat{\lambda}}{2} + \frac{j}{2})\phi - \frac{3\hat{C}j}{8}|\phi|^2\phi + \frac{J}{2} + \frac{j\tilde{A}}{2} + \frac{\hat{B}j}{2}e^{j\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}} = 0, \tag{11}
$$

where

$$
\begin{aligned}
J &= \frac{\hat{A}\omega_{20}\omega_1}{\pi(\omega_1^2 - \omega_{20}^2)}(\cos\,(\frac{\omega_1}{\omega_{20}}2\pi) - 1), \\
\tilde{A} &= \frac{\hat{A}\omega_{20}^2}{\pi(\omega_1^2 - \omega_{20}^2)}\sin\,(\frac{\omega_1}{\omega_{20}}2\pi),
\end{aligned} \tag{12}
$$

that has a solution $\phi(t)$ that remains close to the solution $\phi_\varepsilon(t)$ of Eq. (10) for a certain period of time, and the same initial conditions.

Since we are interested in the amplitude of the oscillations of the nonlinear attachment, we use the polar representation of the complex number $\varphi = N(t)e^{jn(t)}$. After separating the real and imaginary parts, we obtain the following two equations for $N$ and $\eta$.

$$\begin{aligned} N' &= -\frac{\hat{\lambda}N}{2} - \frac{J}{2}\cos(\eta) - \frac{\tilde{A}}{2}\sin(\eta) + \frac{\hat{B}}{2}\sin(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t} - \eta), \\ N\eta' &= -\frac{N}{2} + \frac{3\hat{C}N^3}{8} + \frac{J}{2}\sin(\eta) - \frac{\tilde{A}}{2}\cos(\eta) - \frac{\hat{B}}{2}\cos(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t} - \eta). \end{aligned} \tag{13}$$

This reduced system will be the basis for the asymptotic analysis that will be carried out in the next sections.

## 3 Multiple scales and singular perturbation analysis

We use the following asymptotic method [1,3–6,12] and the multiple scales analysis

$$\begin{aligned} N(t) &= N(t_0,t_1,\ldots) \\ &= N_0(t_0,t_1,\ldots) + \varepsilon N_1(t_0,t_1,\ldots) + O(\varepsilon), \\ \eta(t) &= \eta(t_0,t_1,\ldots) \\ &= \eta_0(t_0,t_1,\ldots) + \varepsilon\eta_1(t_0,t_1,\ldots) + O(\varepsilon), \end{aligned} \tag{14}$$

where $t_0 = \hat{t}$ and $t_1 = \varepsilon\hat{t}$. By keeping $O(1)$ terms in Eqs. (13) for $N$ and $\eta$ we derive the equations

$$\begin{aligned} \frac{\partial N_0}{\partial t_0} + \frac{\hat{\lambda}N_0}{2} + \frac{J}{2}\cos(\eta_0) + \frac{\tilde{A}}{2}\sin(\eta_0) - \frac{\hat{B}}{2}\sin(\varepsilon\bar{B}\hat{t} - \eta_0) &= 0, \\ N_0\frac{\partial \eta_0}{\partial t_0} + \frac{N_0}{2} - \frac{3\hat{C}N_0^3}{8} - \frac{J}{2}\sin(\eta_0) + \frac{\tilde{A}}{2}\cos(\eta_0) + \frac{\hat{B}}{2}\cos(\varepsilon\bar{B}\hat{t} - \eta_0) &= 0. \end{aligned} \tag{15}$$

To study the steady state dynamics of the above system, in terms of the fast time scale $t_0$, we examine the limit of the dynamics as $t_0 \to \infty$ and impose the conditions $\partial\hat{N}_0/\partial t_0 = 0$, $\partial n_0/\partial t_0 = 0$. This will provide us with the long-term behavior of the dynamics in the limit of large values of the fast time scale. Then, from Eqs. (15) we find

$$\begin{aligned} \frac{\hat{\lambda}\hat{N}_0}{2} &= -\frac{J}{2}\cos(\hat{\eta}_0) - \frac{\tilde{A}}{2}\sin(\hat{\eta}_0) + \frac{\hat{B}}{2}\sin(\varepsilon\bar{B}\hat{t} - \hat{\eta}_0) \equiv \bar{\Sigma}_1, \\ \frac{\hat{N}_0}{2} - \frac{3\hat{C}\hat{N}_0^3}{8} &= \frac{J}{2}\sin(\hat{\eta}_0) - \frac{\tilde{A}}{2}\cos(\hat{\eta}_0) - \frac{\hat{B}}{2}\cos(\varepsilon\bar{B}\hat{t} - \hat{\eta}_0) \equiv \bar{\Sigma}_2. \end{aligned} \tag{16}$$

By manipulating expressions (16) we derive the steady state phase

$$\cos\hat{\eta}_0 = \frac{(\tilde{A} + \hat{B}\cos(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}))(\frac{3C\hat{N}_0^3}{4} - \hat{N}_0) + \hat{\lambda}\hat{N}_0(\hat{B}\sin(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}) - J)}{J^2 + \tilde{A}^2 + 2\hat{A}\hat{B}\cos(\frac{\varepsilon B}{\omega_{20}}\hat{t}) + \hat{B}^2 - 2J\hat{B}\sin(\frac{\varepsilon B}{\omega_{20}}\hat{t})}, \tag{17}$$

with the steady state amplitude given by

$$\hat{N}_0^6 - \frac{8}{3\hat{C}}\hat{N}_0^4 + \frac{16(\hat{\lambda}^2+1)}{9\hat{C}^2}\hat{N}_0^2 = \frac{16}{9\hat{C}^2}(\tilde{A}^2 + 2\hat{A}\hat{B}\cos(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}) + \hat{B}^2 + J^2 - 2J\hat{B}\sin(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t})). \tag{18}$$

Equations (17) and (18) represent the slow invariant manifold (SIM) of the dynamics of (7) [1,5] in terms of the fast time scale (note that the slow time scale still appears in coefficients of Eq. (18)).

In order to study the bifurcations of the SIM we reconsider Eqs. (16). After taking the squares of both equations, adding them and introducing the new variable $z = 1 - \frac{3\hat{C}}{4}\hat{N}_0^2$ we derive the equation

$$P(z) = z^3 - z^2 + \hat{\lambda}^2 z - \hat{\lambda}^2 + \frac{3\hat{C}}{4}\bar{\Sigma} = 0, \tag{19}$$

where

$$\bar{\Sigma} = \tilde{A}^2 + 2\tilde{A}\hat{B}\cos(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}) + \hat{B}^2 + J^2 - 2J\hat{B}\sin(\frac{\varepsilon\bar{B}}{\omega_{20}}\hat{t}).$$

From the derivative of the above cubic polynomial

$$P'(z) = 3z^2 - 2z + \hat{\lambda}^2, \tag{20}$$

we see that, for $\hat{\lambda} > 1/\sqrt{3}$, $P(z)$ is a monotone increasing function and therefore has only one real root, while for $\hat{\lambda} < 1/\sqrt{3}$, $P'(z)$ equals zero for $z = (1 \pm \sqrt{1-3\hat{\lambda}^2})/3$, and depending on the value of $\bar{\Sigma}$ the polynomial $P(z)$ may have one or three real roots (Figure 1). When

$$(1-3\hat{\lambda}^2)^{\frac{3}{2}} > |1 + 9\hat{\lambda}^2 - \frac{81C}{8}\bar{\Sigma}|, \tag{21}$$

it has three real roots, otherwise only one. The roots $\hat{N}_0^2 = \frac{4}{3\hat{C}}(1-z_0)$ are positive. We note that when

$$|1 + 9\hat{\lambda}^2 - \frac{81C}{8}\bar{\Sigma}| = (1-3\hat{\lambda}^2)^{\frac{3}{2}}, \tag{22}$$

for a certain value of time, two of the three real roots are equal and saddle-node bifurcations occur.

The above analysis with multiple scales is equivalent to singularity analysis by taking the slow time $\hat{t} \to \varepsilon\bar{B}\hat{t}$. Then system (13) becomes

$$\begin{aligned} \varepsilon N' &= f(N,n,\hat{t}), \\ \varepsilon N n' &= g(N,n,\hat{t}) \end{aligned} \tag{23}$$

and at the singular limit ($\varepsilon = 0$) we have $f = g = 0$ which is exactly the SIM of the system. Tichonov's theorem [9, 13] guarantees that, when the roots of $f = g = 0$ are isolated and at the same time there exist stable solutions of

$$\begin{aligned} \frac{dN}{d\tau} &= f(N,n,\hat{t}), \\ N\frac{dn}{d\tau} &= g(N,n,\hat{t}), \end{aligned} \tag{24}$$

with $\hat{t}$ considered as a parameter, then $N(\hat{t}) \to \hat{N}_0(\hat{t})$, $\eta(\hat{t}) \to \hat{\eta}_0(\hat{t})$ as $\varepsilon \to 0$.

In order to find when the SIM is stable, according to Tichonov's theorem, we find the linear stability of the roots of Eqs. (16) when they are isolated. In the case when Eq. (22) holds the two real roots are not isolated and the time when the bifurcation occurs does not fall under Tichonov's theorem. When the solutions are away from the bifurcation point the matrix of the linearized system around the solution of the SIM is

$$\begin{pmatrix} -\frac{\hat{\lambda}}{2} & \frac{J}{2}\sin(\hat{\eta}_0)-\frac{\tilde{A}}{2}\cos(\hat{\eta}_0)-\frac{\hat{B}}{2}\cos(\varepsilon\bar{B}\hat{t}-\hat{\eta}_0) \\ -\frac{1}{2}+\frac{9\hat{C}}{8}\hat{N}_0^2 & \frac{J}{2}\cos(\hat{\eta}_0)+\frac{\tilde{A}}{2}\sin(\hat{\eta}_0)-\frac{\hat{B}}{2}\sin(\varepsilon\bar{B}\hat{t}-\hat{\eta}_0) \end{pmatrix}$$

where $\hat{N}_0$ and $\hat{\eta}_0$ are the roots of Eq. (16).

Since $\partial\bar{\Sigma}_1/\partial\hat{\eta}_0=\bar{\Sigma}_2, \partial\bar{\Sigma}_2/\partial\hat{\eta}_0=-\bar{\Sigma}_1$ the characteristic polynomial of the linearized system becomes

$$\mu^2+(\bar{\Sigma}_1+\frac{\hat{\lambda}}{2})\mu+\frac{\hat{\lambda}}{2}\bar{\Sigma}_1+(\frac{1}{2}-\frac{9\hat{C}}{8}N_0^2)\bar{\Sigma}_2=0, \tag{25}$$

where $\mu$ are the eigenvalues of the above matrix.

When

$$\bar{\Sigma}_1+\frac{\hat{\lambda}}{2}>0, \tag{26}$$

and

$$\hat{\lambda}\bar{\Sigma}_1+(1-\frac{9\hat{C}}{4}\hat{N}_0^2)\bar{\Sigma}_2>0, \tag{27}$$

the two roots $\mu_1,\mu_2$ of the characteristic polynomial have negative real parts. Using Eq. (16) and the variable $z$, equation (27) becomes

$$3z^2-2z+\hat{\lambda}^2>0 \tag{28}$$

which is the derivative of $P(z)$, calculated at the roots of the defining polynomial of the SIM (19). For $\lambda>1/\sqrt{3}, P'(z)$ is greater than zero for all z and therefore the SIM is stable. For $\hat{\lambda}<1/\sqrt{3}, P'(z)$ is greater than zero outside its roots which are at the same time the maximum and minimum of the polynomial (19). Therefore the solution of $P(z)$, that lies between the minimum and the maximum is unstable, while the lower and the higher roots (when they exist) are stable. Therefore for $\bar{\Sigma}<\frac{8}{81C}((1+9\hat{\lambda}^2)-(1-3\hat{\lambda}^2)^{\frac{3}{2}})$ only the lower solution exists and is stable. The solutions $N(\hat{t}),\eta(\hat{t})$ of (13) tend to it. As time increases and the above inequality is not satisfied, two more roots appear through a saddle-node bifurcation and again disappear, when $\bar{\Sigma}>\frac{8}{81C}((1+9\hat{\lambda}^2)+(1-3\hat{\lambda}^2)^{\frac{3}{2}}$, and for this time interval only the higher root exists.

## 4 Behavior of the solutions and numerical results

Since we have found that, the steady state solutions that are lower or higher than the minimum and maximum of the polynomial $P(z)$, are linearly asymptotically stable, i.e. hyperbolic points, they preserve their stability in the nonlinear system and have a basin of attraction for the interval of time that they exist.

According to Tichonov's theorem the orbits of Eq. (13) tend to these stable branches of the SIM for the above interval of time. Therefore we perform numerical integration of Eq. (13) for

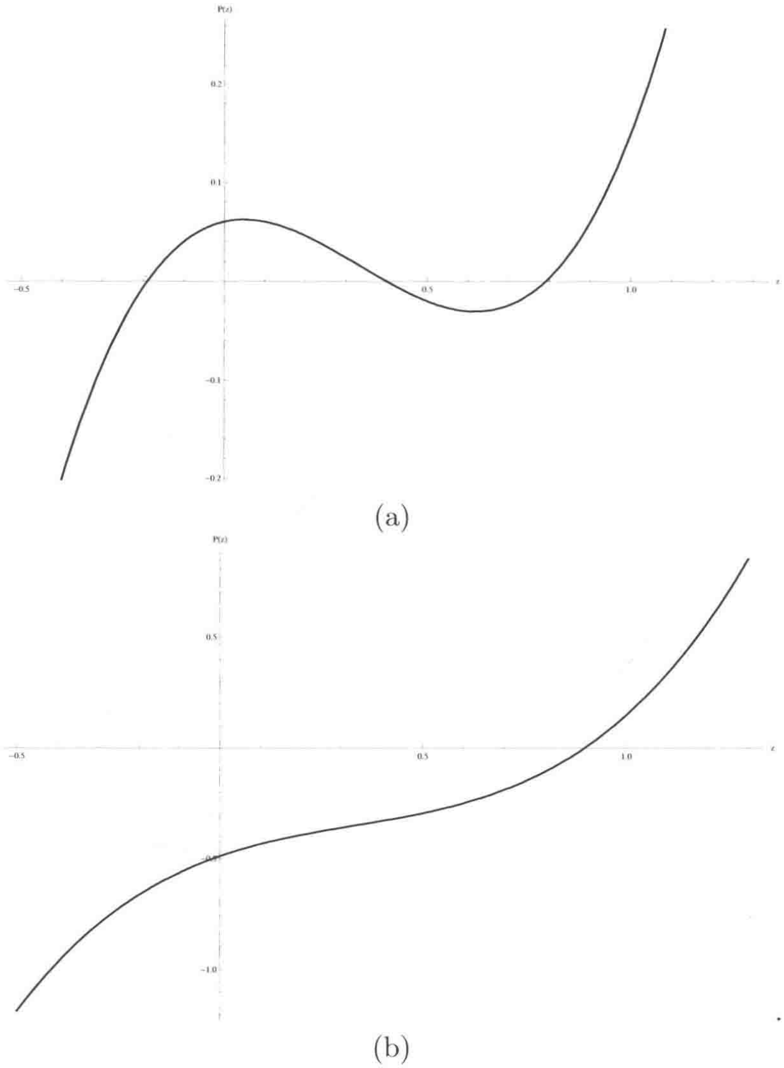



**Fig. 1** The polynomial $P(z)$ for different values of $\hat{\lambda}$:(a) $\hat{\lambda} < \sqrt{1/3}$, (b) $\hat{\lambda} > \sqrt{1/3}$.

different values of the parameters $\hat{A}, \hat{B}, \varepsilon, \hat{\lambda}$ and $\hat{C} = 2$ in order to study what is the relative position of the steady state amplitude after some time in comparison with the position of the SIM.

First we consider the case when the SIM has a single branch. The conditions for this case are:
(i) $\hat{\lambda} > \frac{1}{\sqrt{3}}$,
or (iia) $\hat{\lambda} < \frac{1}{\sqrt{3}}$ and $\bar{\Sigma} < \frac{8}{81\hat{C}}((1+9\hat{\lambda}^2) - (1-3\hat{\lambda}^2)^{\frac{3}{2}})$, for all time,
or (iib) $\hat{\lambda} < \frac{1}{\sqrt{3}}$ and $\bar{\Sigma} > \frac{8}{81\hat{C}}((1+9\hat{\lambda}^2) + (1-3\hat{\lambda}^2)^{\frac{3}{2}})$, for all time.

From the above theoretical analysis, we expect that, for these conditions since we have no bifurcations, the orbits tend to the stable SIM. This is confirmed by direct numerical simulation of systems (13) which is depicted in Fig. 2 and compared to the asymptotic solution predicted by the SIM (18) or (19).

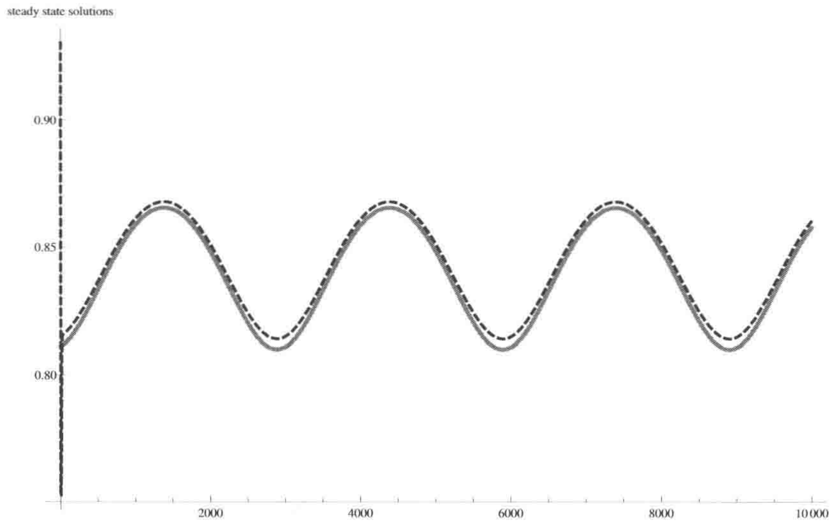



**Fig. 2** $\hat{A} = 0.0748516, \hat{B} = 0.674394, J = 0.00666891$, $\hat{C} = 2, \varepsilon = 0.01, \hat{\lambda} = 0.8$ (gray solid line: SIM, black dashed line: $N(t)$ of (13)).

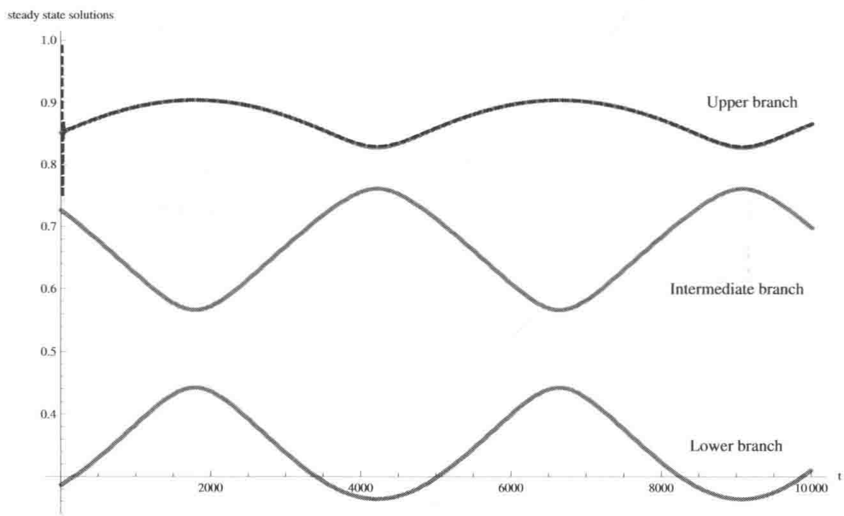



**Fig. 3** $\hat{A} = 0.129, \hat{B} = 0.2956, J = 0.033, \hat{C} = 2$, $\varepsilon = 0.01, \hat{\lambda} = 0.3$ (gray solid line: SIM, black dashed line: $N(t)$ of (13)).

We now consider the second case when the SIM has three branches, two of which are stable and one is unstable. The condition for this case is:

$$\hat{\lambda} < \frac{1}{\sqrt{3}}, \quad \frac{8}{81\hat{C}}((1+9\hat{\lambda}^2) - (1-3\hat{\lambda}^2)^{\frac{3}{2}}) < \bar{\Sigma} < \frac{8}{81\hat{C}}((1+9\hat{\lambda}^2) + (1-3\hat{\lambda}^2)^{\frac{3}{2}}).$$

In this case we also have no bifurcations, and for all time the SIM possesses three branches (Fig. 3). For different initial conditions the orbits of (13) either tend to the upper or to the lower stable branches of the SIM.

The third case is when relaxation oscillations occur, with the dynamics making transitions between the two stable branches of the SIM. In this case bifurcations occur as the orbits of the

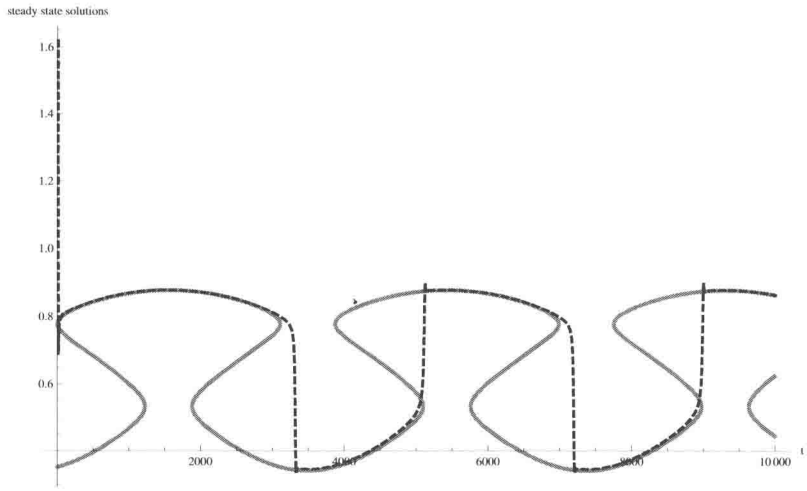



**Fig. 4** $\hat{A} = 0.091, \hat{B} = 0.3463, J = 0.0189, \hat{C} = 2, \varepsilon = 0.01, \hat{\lambda} = 0.4$ (gray solid line: SIM, black dashed line: $N(t)$ of (13)).

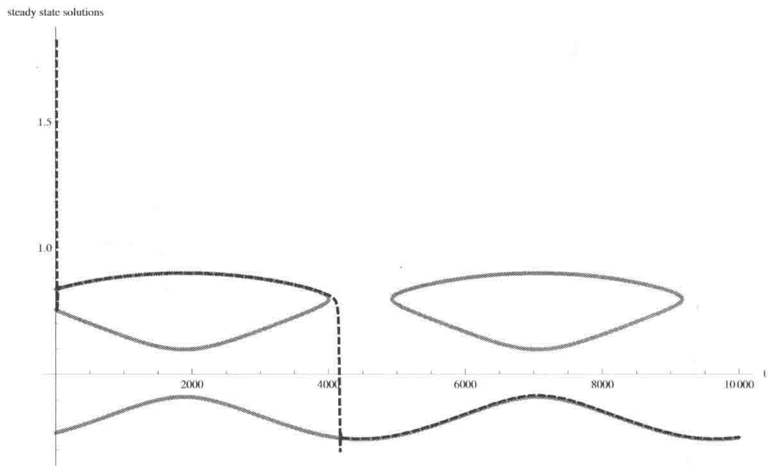



**Fig. 5** $\hat{A} = 0.139, \hat{B} = 0.283, J = 0.0369, \hat{C} = 2, \varepsilon = 0.01, \hat{\lambda} = 0.3$ (gray solid line: SIM, black dashed line: $N(t)$ of (13)).

dynamical system undergo transitions between all branches of the SIM. For a certain period of the slow time scale only the upper branch exists. Then, after a saddle-node bifurcation, two additional branches appear, the one stable and the other unstable. This phenomenon takes place for a period of the slow time scale that depends on the parameters $\hat{A}, \hat{B}, \hat{\lambda}$ and $\hat{C}$. Then the stable upper branch and unstable branch of the SIM coalesce through saddle-node bifurcations. Therefore, the orbits of Eq. (13) that are initially attracted to the upper branch of the SIM, make a sudden transition (jump) to the lower branch producing relaxation oscillations (Fig. 4). After sufficient increase of the slow time scale the upper branch reappears and the lower one disappears again through a saddle node bifurcation. This behavior is repeated periodically and gives rise to sustained relaxation oscillations in the dynamics. These relaxation oscillations indicate strong transient energy transfer in our reduced system, from the system of linear coupled oscillators to the essentially nonlinear attachment.

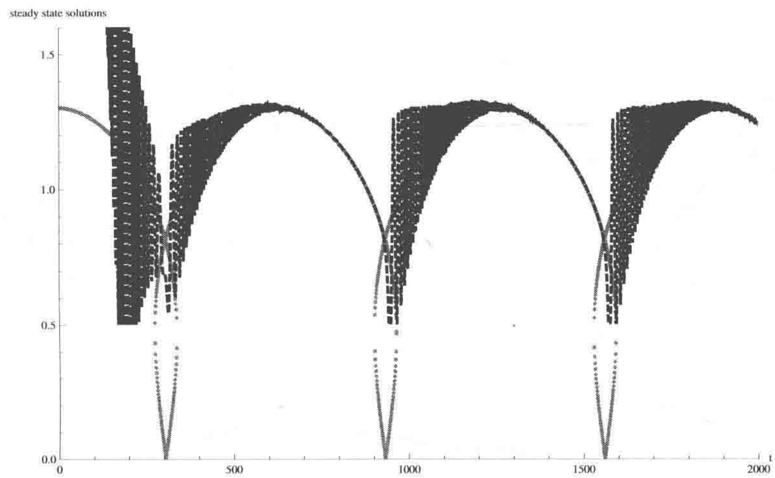



**Fig. 6** $\tilde{A}=1, \hat{B}=1, J=0.1, \hat{C}=2, \varepsilon=0.01, \hat{\lambda}=0.023, \omega_{20}=1, \varepsilon\bar{B}=0.01$ (gray solid line: SIM, black dashed line: $N(t)$ of (13)).

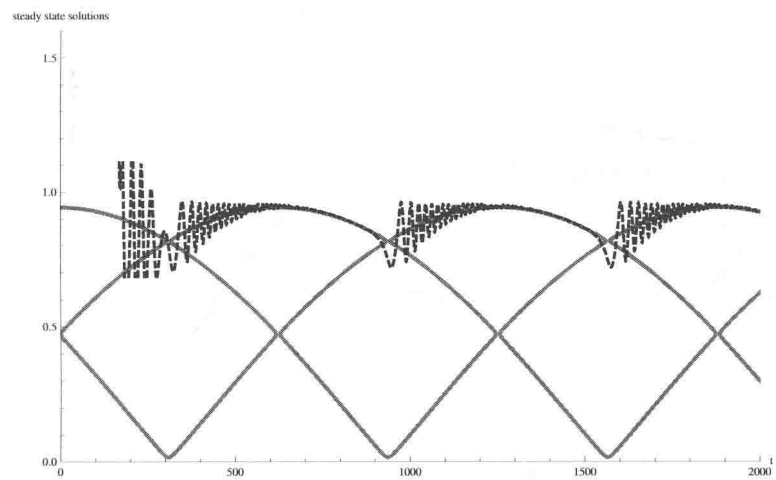
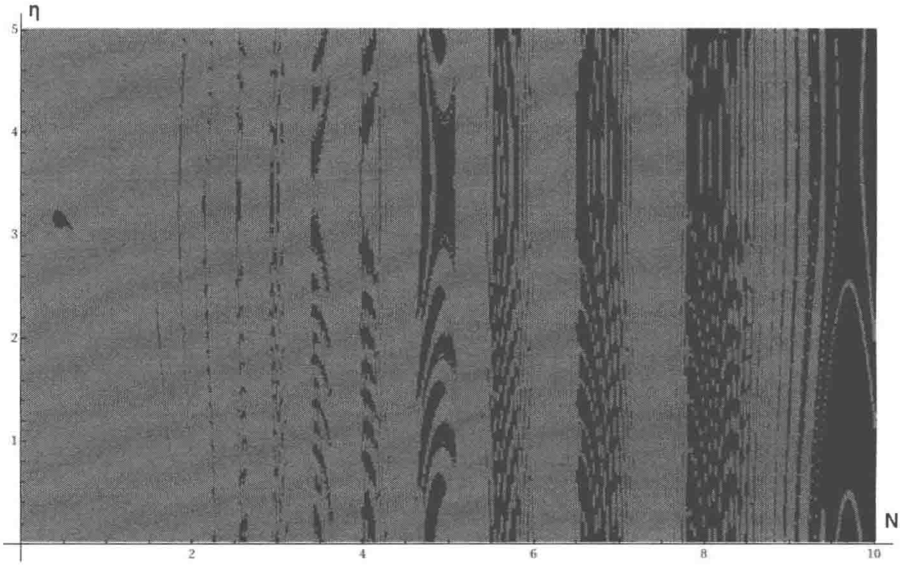


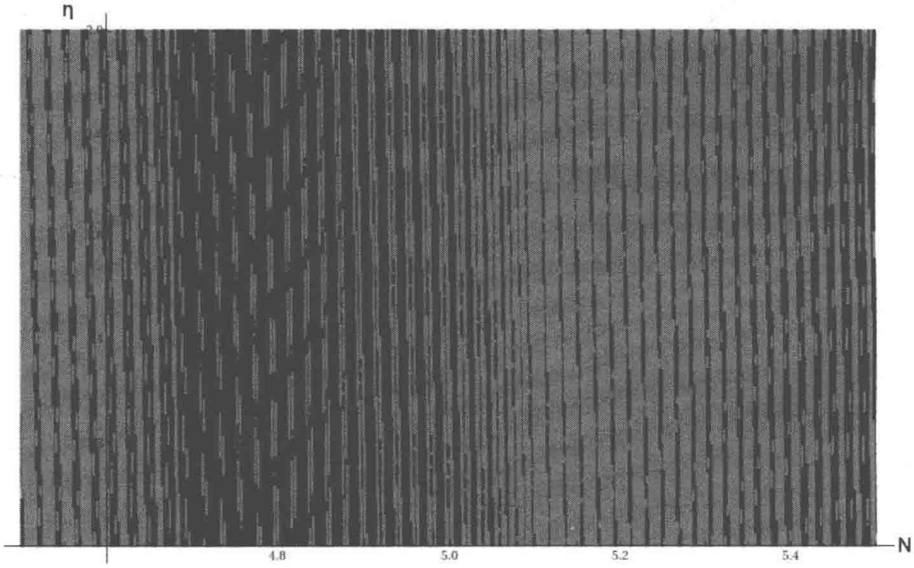

**Fig. 7** $\tilde{A}=0.165319, \hat{B}=0.149, J=0.009289, \hat{C}=2, \varepsilon=0.01, \hat{\lambda}=0.023, \omega_{20}=1, \varepsilon\bar{B}=0.01$ (gray line: SIM, black dashed line: $N(t)$ of (13)).

In the fourth case bifurcations can occur either, between the upper stable branch of the SIM and the unstable branch, or only between the lower stable branch of the SIM and the unstable branch, resulting into the disappearance of a pair of stable-unstable branches of the SIM for a certain period of the slow time scale. From the analysis of Section 3 we expect that the corresponding orbit will be captured in the persisting stable branch of the SIM and this is confirmed by the numerical simulations depicted in Fig. 5.

The fifth case corresponds to the excitation of periodic or non periodic orbits for certain initial conditions. For sufficiently small values of damping $\lambda$ and when there is a crossing of one of the stable branches of the SIM with the unstable branch, but no coalescence, the orbits of (13) may oscillate rapidly, while approaching the stable branch of the SIM, and follow it until the time where we have the crossing (but not a disappearance) of the stable and unstable branches of the SIM. Then an orbit is excited and the entire phenomenon it repeated (Fig. 6).




(a)




(b)

**Fig. 8** Basin of attractions of the stable branches of the SIM of system(13) for $\tilde{A} = 0.165319, \hat{B} = 0.149, J = 0.009289, \hat{C} = 2, \varepsilon = 0.01, \hat{\lambda} = 0.023, \omega_{20} = 1, \varepsilon\bar{B} = 0.01$: (a) $N = (0,10), \eta = (0,5)$, (b)$N = (4.5,5.5), \eta = (1,2)$.

Finally, there is a sixth case where the dynamics of the nonlinear attachment is chaotic, and there is a complicated structure of the basins of attraction of the resulting response. Indeed, for small values of $\lambda$ and when the condition $\frac{8}{81\hat{C}}((1+9\hat{\lambda}^2)-(1-3\hat{\lambda}^2)^{\frac{3}{2}}) \leq \bar{\Sigma} \leq \frac{8}{81\hat{C}}((1+9\hat{\lambda}^2)+(1-3\hat{\lambda}^2)^{\frac{3}{2}})$ is satisfied, there exist always two stable and one unstable branches of the SIM that also cross each other (see Fig. 7). For different initial conditions most of the orbits of (13) either tend to the upper stable or to the lower stable branch of the SIM, following the procedure described in the fifth case, and the basins of attraction of the two stable branches of the SIM have a complicated structure with possible self similarity (Fig. 8). The orbits in the boundary of the two basins of attraction possibly behave in a chaotic way.

## 5 Conclusions

The study of a three degree of freedom dissipative system of linear coupled oscillators with an essentially nonlinear attachment was performed by computing the Slow Invariant Manifold (SIM) of a reduced non-autonomous second order differential equation. And study the stability and bifurcations of this manifold as the system parameters vary.

Depending on the parameters of the system, we have shown through dynamical analysis and Tikhonov's theorem, that the SIM can either have one branch that is stable, or three branches, two which are stable and one unstable. For relatively large values of the damping parameter, the structure of the SIM is simple and the orbits of the system are attracted by it.

The interplay between the stable and the unstable branches of the SIM produces interesting dynamical phenomena such as orbit captures, relaxation oscillations, excitations of periodic orbits and chaotic orbits with complex basins of attraction.

Orbit capture occurs when there is one or two persisting stable branches of the SIM (i.e., stable SIM branches that are not eliminated through bifurcations with progressing slow time). In the case when a single persisting stable branch of the SIM exists, it seems to be globally attractive and the long term behavior of the orbits is simple, since they simply tend to it with increasing time. In the case where one of the stable branches persists and the other two branches bifurcate (e.g., Figure 5) the orbit always relaxes to the persistent stable branch. This behavior can be predicted in terms of the parameters defining the SIM $\hat{A}, \hat{B}, \hat{\lambda}$ and $\hat{C}$ in Eqs. (18) and (19).

Relaxation oscillations occur when we have bifurcations of all three branches of the SIM and this type of dynamics is related to strong transient energy transfer of the reduced system (7), from the linear oscillators to the essentially nonlinear attachment.

The behavior described in the fifth case, where there is excitation of a periodic orbit, and in the sixth case, with the complex structure of the basins of attraction, may relate to transverse homoclinic intersections of the stable and unstable manifolds of the saddle type branch of the SIM. Further investigation of the above phenomena is needed both numerically and analytically in the future.

The exact conditions required for the SIM to have various topologies were found in this work analytically. This allows us to predict the values of the system parameters for which the dynamics have a certain type of behavior. Therefore a control of the evolution of the dynamics of our reduced system, that approximates the model of a structure of linear system with a nonlinear attachment, is achieved.

## References

[1] Gendelman, O.V., Starosvetsky, Y., and Feldman, M. (2008), Attractors of harmonically forced linear oscillator with attached nonlinear energy sink I: Description of response regimes, *Nonlinear Dynamics*, **51**, 31–46.

[2] Gendelman, O., Manevitch, L.I., and Vakakis, A.F. (2001), MCloskey R.,Energy Pumping in Nonlinear Mechanical Oscillators: Part I, Dynamics of the Underlying Hamiltonian Systems, *Journal of Applied Mechanics*, **68**, 34–41.