THE THEORY OF DETERMINANTS IN THE HISTORICAL ORDER OF DEVELOPMENT

Part I. GENERAL DETERMINANTS UP TO 1841
Part II. SPECIAL DETERMINANTS UP TO 1841

BY

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PREFACE.

THE main object of this work and the contents of it will be found specified in the Introductory Chapter. It is intended for the student who aims at acquiring such a knowledge as can only be got by a study of the subject in the historical order of its development, for the investigator who is specially interested in this branch of mathematics and wishes to become acquainted with the various lines of attack opened up by previous workers, and for the general working mathematician who requires guide-books and books of reference concerning special domains.

T. M.

CAPETOWN, SOUTH AFRICA,
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INTRODUCTION.

THE way in which the material for a history of the theory of Determinants has been accumulated is quite similar to that which has been observed in the case of other branches of science. In the middle of the eighteenth century one of the independent discoverers of the fundamental idea, viz., CRAMER, was fortunate enough to attract attention to it, and in time it became the common property of mathematicians in France and elsewhere. As it slowly spread it naturally also received accretions and developments, and of the dozen or so of writers who thus handled it in the sixty years that followed Cramer's publication there were of course a few who by a more or less casual reference kept alive the memory of some of their predecessors. It was then taken up by CAUCHY, and, thanks to the prestige of his name and to the inherent excellence of his extensive monograph, its position as a theory of importance became more firmly assured. The thirty years that followed Cauchy's memoir resembled the sixty that preceded it, save that the number of contributors was considerably larger. Then another great analyst, JACOBI, the most noteworthy of those contributors, produced in Germany a monograph similar in extent and value to Cauchy's, and the importance of the subject in the eyes of mathematicians became still more enhanced. As a consequence, the single decade following gave rise to quite as many new contributions as the preceding three decades had done, and closed with the appearance of the first separately published elementary treatise on the subject, viz., SPOTTISWOODE'S. The preface to this contains the first notable historical sketch of the theory, and includes references to the writings of twelve outstanding mathematicians, beginning with Cramer (1750) and ending with the author's own contemporaries, Cayley, Sylvester and Hermite. In the same year (1850) there also occurred something out of the ordinary, for the correspondence between Leibnitz and the Marquis de l'Hôpital having been published from manuscripts in the Royal Library at Hanover, the striking discovery was made that more than half-a-century before Cramer's time the fundamental idea of determinants had been clear to LEIBNITZ, and had been expounded with considerable fulness by him in a letter to his friend. So strongly attractive had the subject now become to mathematicians that in the single year succeeding the publication of Spottiswoode's short treatise a greater number of separate contributions to the theory made their appearance than in the whole sixty-year period from Cramer to Cauchy. The wants of students everywhere had to be attended to: a second edition of Spottiswoode was consequently prepared for *Crelle's Journal* in 1853; a text-book by Brioschi was published at Pavia in 1854; French and German translations of Brioschi in 1856; and an elementary exposition by Bellavitis in 1857. So far as historical material is concerned, the last-mentioned work was of little account; that of Brioschi resembled Spottiswoode's, the number of references, however, being greater. Of quite a different character was the text-book by BALTZER, which was published at Leipzig the year after the German translation of Brioschi had appeared at Berlin, an important part of the new author's plan being to deal methodically with the history of the subject by means of footnotes. On the enunciation of almost every theorem a note with historical references was added at the foot of the page, the result being that in the portion (thirty-four pages) devoted expressly to the pure theory of determinants about as many separate writings are referred to as there are pages. This was a marked advance, and although during the next twenty years the publication of text-books became more frequent—in fact, if we include those of every language and of every scope, we shall find an average of about one per year—Baltzer's dominated the field; enlarged editions of it appeared in 1864, 1870, and 1875, and the historical notes grew correspondingly in number. Of the other text-books only one, Günther's, which was published in 1875, sought to follow the historical line taken by Baltzer and to add to the supply of material. Then in 1876 another new departure took place, this being the year in which the first writings were published which dealt with the history alone, the one being an academic thesis by E. J. Mellberg printed at Helsingfors, and the other a memoir presented by F. J. Studnička to the Bohemian Society of Sciences.

About this time, while engaged in writing my own so-called "Treatise on the Theory of Determinants," I had occasion to look into the question of the authorship and history of the various theorems, and I was reluctantly forced to the conclusion that much inaccurate statement prevailed in regard to such matters and that the whole subject was worthy of serious investigation. A resolution was accordingly taken to set about collecting the titles of all the writings which had appeared on the theory up to the end of 1880. The task was not an easy one, as will readily be understood by those who know how scanty and defective are the bibliographical aids at the disposal of mathematicians, and how often the titles given by investigators to their memoirs are imperfect and even misleading in regard to the nature of the contents. The outcome of the search was published in 1881 in the October number of the *Quarterly Journal of Mathematics* (vol. xviii. pp. 110–149) under the title of "A List of Writings on Determinants." It contained 589 entries arranged in chronological order. Some three or four years afterwards, when there had been time to test the completeness of the earlier portion of the list, the writings included in it were taken up in historical succession and suitable abstracts or reviews of them made for publication in the *Proceedings of the Royal Society of Edinburgh*; the first contribution of this kind was presented to the Society in the beginning of the year 1886. At the same time there was being prepared an additional list of writings containing omitted titles, 84 in number, belonging to the period of the first list, and 176 titles belonging to the further period 1881–1885. This second list appeared in 1886 in the June number of the *Quarterly Journal of Mathematics* (vol. xxi. pp. 299–320). In 1890 a collection was made of the contributions, just mentioned, which had up to that date been printed in the Edinburgh *Proceedings*, and with the consent of the Society was published separately. Unfortunately in that year all this train of work had to be laid aside on account of the pressure of official duties, and ten years elapsed before it could be resumed. It was thus not until March 1900 that a second series of analytic abstracts began to appear in the Edinburgh *Proceedings*, and that the preparation of a third list of writings was methodically undertaken. The period to be covered by this list was the fifteen years 1886–1900; and as the number of writers interested in the subject had in these years continued to increase, and as closer examination of the literature of the previous periods had led to new finds, the resulting compilation was more extensive than the first two put together. It was presented to the South African Association for the Advancement of Science at its inaugural meeting in April 1903 and was published in the Report; it is also to be found in the *Quarterly Journal of Mathematics* for December 1904 and February 1905 (vol. xxxvi. pp. 171–267). The number of titles in the three lists is about 1740; they furnish, it is hoped, an almost complete guide to the literature of the theory of determinants from the earliest times to the close of the nineteenth century.

From these later labours it became manifest that it was undesirable in the way of separate publication to issue merely another volume as a continuation of, and similar to, that of the year 1900. The better course clearly was to reproduce the material of that volume along with the intercalations necessitated in it by the existence of subsequently discovered papers, and to follow this up in such a way as to give finally within the compass of a reasonably sized volume a full history of the subject in all its branches up to about the middle of the nineteenth century. This is what is here attempted.

The plan followed is not to give one connected history of determinants as a whole, but to give separately the history of each of the sections into which the subject has been divided, viz., to deal with determinants in general, and thereafter in order with the various special forms. This will not only tend to smoothness in the narrative by doing away with the necessity of frequent harkings back, but it will also be of material importance to investigators who may wish to find out what has already been done in advancing any particular department of the subject. To this end, also, each new result as it appears will be numbered in Roman figures; and if the same result be obtained in a different way, or be generalised, by a subsequent worker, it will be marked among the contributions of the latter with the same Roman figures, followed by an Arabic numeral. Thus the theorem regarding the effect of the transposition of two rows of a determinant will be found under Vandermonde, marked with the number XI., and the information intended thus to be conveyed is that in the order of discovery the said theorem was the *eleventh* noteworthy result obtained: while the mark XI. 2, which occurs under Laplace, is meant to show that the theorem was not then heard of for the first time, but that Laplace contributed something additional to our knowledge of it. In this way any reader who will take the trouble to look up the sequence XI., XI. 2, XI. 3, &c., may be certain, it is hoped, of obtaining the full history of the theorem in question.

The early foreshadowings of a new domain of science, and tentative gropings at a theory of it, are so difficult for the historian to represent without either conveying too much or too little, that the only satisfactory way of dealing with a subject in its earliest stages seems to be to reproduce the exact words of the authors where essential parts of the theory are concerned. This I have resolved to do, although to some it may have the effect of rendering the account at the commencement somewhat dry and forbidding.

CHAPTER II.

DETERMINANTS IN GENERAL, FROM THE YEAR 1693 TO 1779.

THE writers here to be dealt with are seven in number, viz., Leibnitz, Fontaine, Cramer, Bézout, Vandermonde, Laplace, Lagrange. Of these the first two exercised no influence on the development of the theory; the real moving spirit was Cramer; Lagrange alone of the others may have been unaffected by this particular part of Cramer's work.

LEIBNITZ (1693).

[Leibnizens mathematische Schriften, herausg. v. C. I. Gerhardt. 1 Abth. ii. pp. 229, 238–240, 245. Berlin, 1850.]

In the fourth letter of the published correspondence between Leibnitz and De L'Hospital, the former incidentally mentions that in his algebraical investigations he occasionally uses numbers instead of letters, treating the numbers however as if they were letters. De L'Hospital, in his reply, refers to this, stating that he has some difficulty in believing that numbers can be as convenient or give as general results as letters. Thereupon Leibnitz, in his next letter (28th April 1693), proceeds with an explanation:—

"Puisque vous dites que vous avés de la peine à croire qu'il soit aussi general et aussi commode de se servir des nombres que des lettres, il faut que je ne me sois pas bien expliqué. On ne sçauroit douter de la generalité en considerant qu'il est permis de se servir de 2, 3, etc., comme d' *a* ou de *b*, pour veu qu'on considere que ce ne sont pas de nombres veritables. Ainsi 2.3 ne signifie point 6 mais autant qu' *ab*. Pour ce qui est de la commodité, il y en a des tres grandes, ce qui fait que je m'en sers souvent, sur tout dans les calculs longs et difficiles ou il est aisé de se tromper. Car outre la commodité de l'épreuve par des nombres, et même par l'abjection du novenaire, j'y trouve un tres grand avantage même pour l'avancement de l'Analyse. Comme c'est une ouverture assez extraordinaire, je n'en ay pas encor parlé à d'autres, mais voicy ce que c'est. Lorsqu'on a besoin de beaucoup de lettres, n'est il pas vray que ces lettres n'expriment point les rapports qu'il y a entre les grandeurs qu'elles signifient, au lieu qu'en me servant des nombres je puis exprimer ce rapport. Par exemple soyent proposées trois equations simples pour deux inconnues à dessein d'oster ces deux inconnues, et cela par un canon general. Je suppose

$$10+11x+12y = 0 \quad (1)$$

$$\text{et} \quad 20+21x+22y = 0 \quad (2)$$

$$\text{et} \quad 30+31x+32y = 0 \quad (3)$$

ou le nombre feint estant de deux characteres, le premier me marque de quelle equation il est, le second me marque à quelle lettre il appartient. Ainsi en calculant on trouve par tout des harmonies qui non seulement nous servent de garans, mais encor nous font entrevoir d'abord des regles ou theoremes. Par exemple ostant premierement y par la premiere et la seconde equation, nous aurons :

$$\begin{matrix} +10\,.\,22+11\,.\,22x \\ -12\,.\,20-12\,.\,21\,.\,. \end{matrix} = 0 \quad (4)^*$$

et par la premiere et troisieme nous aurons :

$$\begin{matrix} +10\,.\,32+11\,.\,32x \\ -12\,.\,30-12\,.\,31\,.\,. \end{matrix} = 0 \quad (5)$$

ou il est aise de connoistre que ces deux equations ne different qu'en ce que le charactere antecedent 2 est changé au charactere antecedent 3. Du reste, dans un même terme d'une même equation les characteres antecedens sont les mêmes, et les characteres posterieurs font une même somme. Il reste maintenant d'oster la lettre x par la quatrieme et cinquieme equation, et pour cet effect nous aurons †

$$\begin{matrix} 1_0\,.\,2_1\,.\,3_2 \\ 1_1\,.\,2_2\,.\,3_0 \\ 1_2\,.\,2_0\,.\,3_1 \end{matrix} = \begin{matrix} 1_0\,.\,2_2\,.\,3_1 \\ 1_1\,.\,2_0\,.\,3_2 \\ 1_2\,.\,2_1\,.\,3_0 \end{matrix}$$

qui est la derniere equation delivrée des deux inconnues qu'on vouloit oster, et qui porte sa preuve avec soy par les harmonies qui se remarquent par tout, et qu'on auroit bien de la peine à decouvrir en

* This is written shortly for $\left.\begin{matrix} +10\,.\,22+11\,.\,22x=0 \\ -12\,.\,20-12\,.\,21x=0 \end{matrix}\right\}.$

† The author here slightly changes his notation. What is meant to be indicated is

$$10.21.32 + 11.22.30 + 12.20.31 = 10.22.31 + 11.20.32 + 12.21.30.$$

employant des lettres *a*, *b*, *c*, sur tout lors que le nombre des lettres et des equations est grand. Une partie du secret de l'analyse consiste dans la caracteristique, c'est à dire dans l'art de bien employer les notes dont on se sert, et vous voyés, Monsieur, par ce petit echantillon, que Viete et des Cartes n'en ont pas encor connu tous les mysteres. En poursuivant tant soit peu ce calcul on viendra à un *theoreme general* pour quelque nombre de lettres et d'equations simples qu'on puisse prendre. Le voicy comme je l'ay trouvé autres fois :

"*Datis aequationibus quotcunque sufficientibus ad tollendas quantitates, quae simplicem gradum non egrediuntur, pro aequatione prodeunte, primo sumendae sunt omnes combinationes possibiles, quas ingreditur una tantum coefficiens uniuscujusque aequationis: secundo, eae combinationes opposita habent signa, si in eodem aequationis prodeuntis latere ponantur, quae habent tot coefficientes communes, quot sunt unitates in numero quantitatum tollendarum unitate minuto: caeterae habent eadem signa.*

"J'avoue que dans ce cas des degrés simples on auroit peut estre decouvert le même theoreme en ne se servant que de lettres à l'ordinaire, mais non pas si aisement, et ces adresses sont encor bien plus necessaires pour decouvrir des theoremes qui servent à oster les inconnues montées à des degrés plus hauts. Par exemple, ... "

It will be seen that what this amounts to is *the formation of a rule for writing out the resultant of a set of linear equations*. When the problem is presented of eliminating x and y from the equations

$$a+bx+cy = 0, \qquad d+ex+fy = 0, \qquad g+hx+ky = 0,$$

Leibnitz in effect says that first of all he prefers to write 10 for a, 11 for b, and so on; that, having done this, he can all the more readily take the next step, viz., forming every possible product whose factors are one coefficient from each equation,* the result being

$$10.21.32, \qquad 10.22.31, \qquad 11.20.32,$$
$$11.22.30, \qquad 12.20.31, \qquad 12.21.30;$$

and that, then, *one* being the number which is less by one than the number of unknowns, he makes those terms different in sign which have only *one* factor in common.

The contributions, therefore, which Leibnitz here makes to algebra may be looked upon as three in number :—

(1) A *new notation*, numerical in character and appearance, for individual members of an arranged group of magnitudes; the two members which constitute the notation being like the

* Of course, this is not exactly what Leibnitz meant to say.

Cartesian co-ordinates of a point in that they denote any one of the said magnitudes by indicating its position in the group. (I.)

(2) A rule for *forming the terms* of the expression which equated to zero is the result of eliminating the unknowns from a set of simple equations. (II.)

(3) A rule for *determining the signs* of the terms in the said result. (III.)

The last of these is manifestly the least satisfactory. In the first place, part of it is awkwardly stated. Making those terms different in sign *which have only as many factors alike as is indicated by the number which is less by one than the number of unknown quantities* is exactly the same as making those terms different in sign *which have only two factors different.* Secondly, in form it is very unpractical. The only methodical way of putting it in use is to select a term and make it positive; then seek out a second term, having all its factors except two the same as those of the first term, and make this second term negative; then seek out a third term, having all its factors except two the same as those of the second term, and make this third term positive; and so on.

Although there is evidence that Leibnitz continued, in his analytical work, to use his new notation for the coefficients of an equation (see Letters xi., xii., xiii. of the said correspondence), and that he thought highly of it (see Letter viii. "chez moi c'est une des meilleures ouvertures en Analyse"), it does not appear that by using it in connection with sets of linear equations, or by any other means, he went further on the way towards the subject with which we are concerned. Moreover, it must be remembered that the little he did effect had no influence on succeeding workers. So far as is known, the passage above quoted from his correspondence with De L'Hospital was not published until 1850. Even for some little time after the date of Gerhardt's publication it escaped observation, Lejeune Dirichlet being the first to note its historical importance. It is true that during his own lifetime, Leibnitz's *use of numbers in place of letters* was made known to the world in the *Acta Eruditorum* of Leipzig for the year 1700 (*Responsio ad Dn. Nic. Fatii Duillerii imputationes,* pp. 189–208); but the particular application of the new symbols which brings them into connection with determinants was not there given.

In a subsequent volume of *Leibnizens mathematische Schriften*,—the third volume of the second Abtheilung,—published at Halle in 1863, the following equivalent of the above 'théorème général' appears (pp. 5–6):—

"Inveni Canonem pro tollendis incognitis quotcunque aequationes non nisi simplici gradu ingredientibus, ponendo aequationum numerum excedere unitate numerum incognitarum. Id ita habet.

Fiant omnes combinationes possibiles literarum coefficientium ita ut nunquam concurrant plures coefficientes ejusdem incognitae et ejusdem aequationis. Hae combinationes affectae signis, ut mox sequetur, componuntur simul, compositumque aequatum nihilo dabit aequationem omnibus incognitis carentem.

Lex signorum haec ist. Uni ex combinationibus assignetur signum pro arbitrio, et caeterae combinationes quae ab hac differunt coefficientibus duabus, quatuor, sex etc. habebunt signum oppositum ipsius signo: quae vero ab hac differunt coefficientibus tribus, quinque, septem etc. habebunt signum idem cum ipsius signo. Ex. gr. sit

$$10+11x+12y=0, \qquad 20+21x+22y=0, \qquad 30+31x+32y=0;$$

fiet
$$+10.21.32-10.22.31-11.20.32$$
$$+11.22.30+12.20.31-12.21.30=0.$$

Coefficientibus eas literas computo, quae sunt nullius incognitorum, ut 10, 20, 30."

Although Gerhardt, the editor, states that the original manuscript of Leibnitz, from which this is taken, bears no date, it is very probable to date farther back than 1693, and not impossible to belong to 1678.*

FONTAINE (1748).

[Mémoires donnés à l'Académie Royale des Sciences, non imprimés dans leurs temps. Par M. Fontaine † de cette Académie. 588 pp. Paris, 1764.]

These memoirs of Fontaine's, sixteen in number, cover a considerable variety of mathematical subjects: it is the seventh of

* See also GERHARDT, K. I., Leibniz über die Determinanten, *Sitzungsb.* ..... *Akad. d. Wiss.* (Berlin), 1891, pp. 407–423.

† The full name is *Alexis Fontaine des Bertins*. The very same collection was issued in 1770 under the less appropriate title *Traité de calcul différentiel et intégral*. Vandermonde is said to have been a pupil of Fontaine's (*v. Nouv. Annales de Math.*, v. p. 155).