ANALOG SIMULATION

SOLUTION OF FIELD PROBLEMS

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PREFACE

The decade following World War II has seen tremendous progress in the development of automatic computers and in their application to engineering problems. The perfection of high-capacity, accurate general-purpose analog and digital computers has opened up to the engineer entirely new avenues of analysis and design and has permitted the treatment of problems of ever-increasing complexity and difficulty.

As both analog and digital equipment became available to industrial and university research organizations, a logical "division of labor" took place between these two major computing methods. Digital computers are generally recognized as the appropriate tool for dealing with problems in which high accuracy is required and in which a large number of items of data must be tabulated and manipulated. Problems in which an accuracy of about 1 per cent is adequate and in which a direct insight into the physics of the system under study is desired are more effectively and conveniently handled by electric or electronic-analog equipment. In this respect analog computers have become particularly useful in dealing with problems involving differential equations.

Differential equations constitute the heart of virtually all branches of engineering, and the description of physical phenomena by ordinary or partial differential equations generally constitutes the first step in the scientific analysis or synthesis of engineering systems. The power and utility of the analog method lie in the fact that the characteristic equations of a wide variety of engineering fields are inherently similar and subject to the same analytical treatment. Therefore, a small number of highly refined computing techniques are effective in solving most of the differential equations encountered in engineering work. The optimum utilization of the analog approach then demands a grasp of the mathematical foundation of engineering systems, as well as a thorough understanding of the capabilities and limitations of available analog equipment and available computing techniques.

An examination of recent technical literature reveals that a comprehensive and effective computer philosophy exists for the solution of engineering problems governed by ordinary differential equations—that is, problems in which the system parameters occur in lumped form, as is the case in dynamic analysis and servomechanisms. On the other hand, the treatment of systems described by partial differential equations, in which the parameters are distributed or continuous in nature, does not seem to have enjoyed such a general approach. A major reason for this disparity lies in the manner in which the analog computing field developed.

The analog treatment of lumped systems had to await the perfection of electronic differential analyzers—relatively large complex general-purpose instruments which first became available commercially about 1948. The requirements of the aircraft industry and competition between several computer manufacturers gave a sudden impetus to a systematic investigation of the application of this new equipment to a wide range of problems. The modern analog approach to ordinary differential equations, therefore, started with the appearance of a powerful new device, followed by a search for new applications for this device.

The effective treatment of distributed systems by means of electric analogs preceded that of lumped systems by at least twenty years. The emphasis, however, was invariably on special-purpose devices and techniques. No organization recognized a financial stake in systematizing and generalizing the methods used to make them applicable outside a narrow field of interest. Electric analogs are now widely used to simulate systems governed by partial differential equations in such diverse fields as petroleum production, irrigation, geophysics, nuclear-reactor design, microwave propagation, and aircraft structures and have achieved great economic importance in these and other areas. An over-all philosophy for handling these problems seems to be lacking, however.

It is the purpose of this text to help fill this gap by providing a comprehensive survey of distributed system analogs together with a concise presentation of the mathematical tools necessary for their optimum utilization. The approach to the subject is a unified one. The mathematical theory linking the characteristic problems of the various fields of engineering is developed early in the text. Subsequent examples of the applications of the electric-analog techniques are then classified and organized according to their mathematical properties rather than the specialized field in which they became popular. It is hoped that such a general approach will stimulate further research and progress in this area along a "broad front" rather than along many independent avenues as has been the case in the past.

The reader is assumed to have a general engineering background of the type generally attained in the first six or seven semesters of an undergraduate engineering curriculum. A reasonable familiarity with ordinary differential equations, as well as some acquaintance with electrical-circuit theory, is required. Additional mathematical and electrical concepts are developed wherever necessary in the text.

The content of the text falls naturally into a general Introduction and three major divisions: 1, The Mathematical Model; 2, Analog-simulation Systems; and 3, Analog Applications. Each of these divisions is further subdivided into several chapters.

The introductory chapter is opened with a general discussion of the occurrence of field problems in engineering work. The typical characteristics of engineering problems are then considered—what is generally given, and what is to be found, i.e., what constitutes a solution. The place of electric-analog methods and their advantages and disadvantagse in engineering projects is then described, followed by a short evaluation of the digital-computer approach to the same problems.

The purpose of Part 1, The Mathematical Model, is not to comprise an extensive mathematical treatise; rather it is to present in a concise and usable form the essence of those mathematical principles and techniques which are of immediate and direct interest in solving partial differential equations by electric analogs. Proofs and lengthy derivations are omitted. Where possible, tabular or graphical summaries are included for easy reference.

Chapter 2 introduces the conventional terminology and notation of partial differential equations and demonstrates how basic physical assumptions such as continuity and conservation lead to the familiar characteristic equations. The physical implications of these equations, the nature of their formulation, and the characteristics of their solution are considered in considerable detail.

Chapter 3, Transformations, shows how the basic equations can be manipulated through appropriate changes of variables to facilitate their solution by analog methods. A brief discussion of the basic coordinate systems (cartesian, cylindrical, spherical) leads to a concise treatment of conformal transformations. A table showing over twenty of the most useful transformations is included. The chapter includes a survey of integral transformation techniques by which partial differential equations can be converted to ordinary differential equations. The principles of duality and superposition and their utilization in analog-simulation work are also discussed.

Chapter 4 is devoted to the important finite-difference approximations. The method of expanding partial differential equations into a set of finite-difference equations is described and summarized in a table listing the approximations of the more important partial differential equations in the three principal coordinate systems. The treatment of boundary and initial conditions and error-reduction techniques are then considered. A discussion of convergence and stability concludes the chapter.

Part 2, Analog-simulation Systems, is a comprehensive study of the various methods of simulating field problems by analogs. The basic operation of each system is described in some detail together with such auxiliary techniques as have been found useful. The basic advantages and limitations of the analog systems are then considered, and sources of error are analyzed in a quantitative manner.

Chapter 5 treats the conductive-solid analogs, particularly the resistance-paper methods. Chapter 6 is devoted to the electrolytic-tank-analog systems. Chapter 7 describes the important resistance-network analyzers, while Chapter 8 includes a detailed treatment of network analyzers containing reactive elements as well as resistors. Particular emphasis is placed on the *RC*-network analogs. Chapter 9 treats the direct applications of electronic differential analyzers to the solution of partial differential equations, as well as their utilization in supplementing conductive-sheet and network analogs. Chapter 10 is a survey of the more important nonelectric-analog systems such as fluid mappers, stretched membranes, and ion-diffusion simulators.

Part 3 illustrates the application of the mathematical techniques of Part 1 and the utilization of the analog systems of Part 2 to the solution of typical engineering problems. These problems are categorized according to their mathematical form. The general nature of each solution is stressed, and a special effort is made to provide a physical insight into the relations between the elements of the analog system and the corresponding elements of the prototype system under study.

In this manner Chapter 11 treats field problems governed by Laplace's, Poisson's, and other elliptical partial differential equations; Chapter 12 illustrates the solution of the diffusion equation and other parabolic equations; while Chapter 13 is devoted to problems governed by the wave equation as well as by other hyperbolic equations. Chapter 14 illustrates solutions of stress problems governed by the fourth-order biharmonic equation.

The material covered in this text can be integrated in an engineering curriculum in a number of ways. It has been used for a number of years as the second half of a one-semester introductory course in analog computation, taught on a senior or first-year graduate level. The first part of this course deals with the solution of ordinary differential equations by means of electronic analog computers, while the second half is devoted to field problems. Chapters 3 and 11 to 14 are usually omitted in such an introductory course. This book has also served as the text for a full semester course dealing entirely with the analog simulation of field problems. In such a course the bulk of the students are graduate students who have completed an introductory course in analog computations.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. What Is a Field? The problems of engineering and applied physics can be classified into two broad categories: those which apply to lumped systems and those which apply to distributed systems. Familiar examples of lumped systems appear in mechanics and in electric-circuit theory. Figures 1.1*a* and *b* illustrate two such lumped circuits.

In the mechanical circuit the spring k and the mass M represent reservoirs of potential and kinetic energy, respectively, while the dashpot D is the dissipative element. Likewise, in the electric circuit, the inductor L and the capacitor C store electric energy while the resistor R dissipates energy. The manner in which these elements react to the application of a force F or a voltage source e determines entirely the behavior or response of the system. It is to be recognized that the lines connecting these elements, perhaps linkages in the mechanical circuit and wires in the electric circuit, have no significance other than to indicate how the elements are interconnected. The lengths of these lines or their arrangement is immaterial, and the circuits could be redrawn in an infinite number of ways and still convey the same information.

Another characteristic of these circuits, and this really constitutes the distinctive feature of lumped systems, is that the physical dimensions and position of the elements are of no direct consequence. As far as mechanical-circuit theory is concerned, the symbol labeled k in the circuit diagram merely represents a functional relationship: namely, that the force through the spring is proportional to a constant times the displacement between its two extremities, and all the relevant properties of the spring are lumped in this designation. A number of assumptions are implicit in this statement. For example, it is presumed that the weight of the spring is negligible. Actually this is not exactly correct, of course, and the upper half of the spring is necessarily elongated a bit more than the lower half, since it is acted upon to a greater extent by the weight of the spring itself. Another assumption made in lumping the characteristics of the spring is that, when a force is applied to one side of the element, this force is felt immediately and in exactly the same form at the other end of the element and everywhere within the element. Evidently if the mass and frictional damping within the spring are of consequence, the transient force at the top of the spring is different from that applied at the bottom.

The lumped approximation of the characteristics of a spring is satisfactory for many problems of dynamics and facilitates the analysis of many mass spring systems without excessive error. This approach is completely inadequate, however, if one is interested in the internal behavior of the spring, that is, how much each segment of the spring is stressed or strained. Then no choice exists but to recognize that each segment of the spring, no matter how small, has the properties of mass and damping as well as those of a pure spring and that all these properties must be

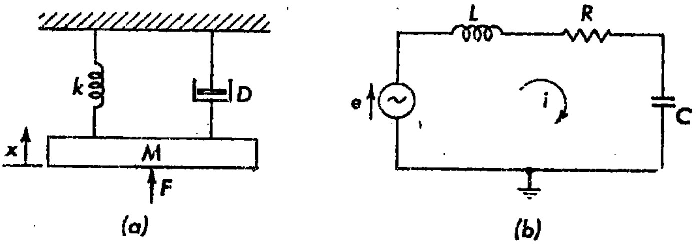

FIG. 1.1. Example of lumped systems.

considered together in studying the behavior of the spring under stress. The convenient idea of a lumped element must be relinquished; a new concept, that of continuous or distributed parameters, must be introduced; and all stresses and strains must be identified as to their *location* along the length of the spring. *The lumped problem has become a field problem.* The distinctive feature of a field problem is that spatial dimensions comprise an integral part of the formulation and solution of the problem. Mathematically speaking, location in space is characterized by additional independent variables, whereas in the lumped problem time is the only independent variable. Some field problems can be formulated completely by referring only to one space dimension. The dynamic problem just discussed is an example of such a one-dimensional system. If the spring, instead of being a conventional coil spring, is rather in the form of an elastic sheet, the horizontal spatial position of a point on the sheet, as well as its vertical position, must be specified in studying the internal strain. The formulation of the field problem then involves two space variables. A spring, shaped in the form of an elastic cylinder of appreciable radius, presents a three-dimensional problem. In each case the n-dimensional space, within which the spring is confined to move, constitutes a *field*.

This transition from a lumped dynamic system to a distributed system has been discussed in considerable detail to demonstrate the basic character of field problems. Precisely the same considerations apply in many other areas of applied physics and technology. In electric circuits, such as the one indicated in Fig. 1.1*b*, the resistor may contain inductance and capacitance distributed along its entire length. The determination of the voltage and current at some point within the resistor, therefore, involves a consideration of the location of that point within the resistor and a study of the nature of its distributed electrical characteristics. The flow of heat through a conductive medium, the flow of fluids through a porous medium or through channels, the propagation of sound and electromagnetic waves through space, the gravitational and magnetic attraction between two bodies—all these phenomena require for their complete specification and study a knowledge of distributed parameters and of the location of the point under study within a region of space—a field—and hence are field problems.

The word "problem" has come up again and again in this discussion. It is appropriate, therefore, that some consideration now be given to the significance of this term in an engineering context.

1.2. What Constitutes an Engineering Problem? The essence of most engineering endeavors is the analysis or synthesis of physical systems. A system is an assemblage of elements ranging in number from one to infinity, in either lumped or continuous form, which reacts to an excitation in a known or predictable manner. For example, the circuits of Figs. 1.1*a* and *b* comprise mechanical and electrical systems, respectively. In field problems, the entire field under consideration is identified as the system and may be a thermal conductor, a bounded region of space in electrostatics, an oil reservoir, or an electric transmission line. Such a system is said to be passive if it contains no internal-energy sources. All the systems just mentioned are *passive*. If internal-energy sources are present, as, for example, a vacuum-tube amplifier in an electric circuit, the system is said to be *active*. In either case, the application of energy to the system brings forth a reaction within the system. In the mechanical circuit, the application of energy may be in the form of forces or velocities at specified points in the circuit. As a result, the elements of the circuit may be set in motion or strained. In the electric circuit, energy may be applied by voltage or current sources, and voltage drops and currents may appear as a result in other parts of the circuit. In any event, a definite cause-effect relationship exists, as determined by the characteristics of the elements. The cause is generally termed the excitation or driving function, and the effect, the response. This relationship is indicated in Fig. 1.2.

If the excitation varies with time, the response is also a function of time, and transient conditions are said to exist. Steady-state or static conditions imply that either the excitation has remained unchanged at all times or enough time has elapsed since any change in excitation so that the response has assumed a constant value. It should be clearly understood that, in Fig. 1.2, "excitation" does not necessarily refer to a single source. For example, voltage and current sources could act simultaneously in various parts of an electric circuit. Likewise, "response" includes the effects at all points in the circuit which are of interest. These may include the place at which the excitation is applied. If a 5-ohm resistor is connected across a 10-volt battery, the battery is the excitation, the resistor is the system, and a current of 2 amp is the steady-state response.

To solve a problem of *analysis* is to determine the response due to a given excitation acting upon a known or fully specified system. In the case of a lumped circuit, the specification of the system involves a concise, possibly but not necessarily mathematical, description of the characteristics of each element and precisely how the elements are interconnected; in addition, the location, magnitude, and transient characteristics of all sources acting on the circuit must be furnished. A similar set of specifications is required in a field-analysis problem. Here, however, the specifications must include both the distributed characteristics of all points within the field and the location of the field boundaries. The space coordinates of all excitations must likewise be specified. These may act along the field boundaries, at specific regions within the field, or they may be distributed in a continuous fashion throughout the field, as, for example, the force of gravity acting uniformly on the distributed mass of a spring.

FIG. 1.2. Causal relationship in physical systems.

The distinction between transient and static conditions is particularly significant in the case of field problems. If the response of the system is a function of time, the problem is called an "initial-value problem." In that case, in order to predict the response of the system, it is necessary to know the response values of the system at some time, conveniently referred to as the "initial instant," as well as all the excitation functions at times subsequent to the initial time. For example, if the force F exciting the spring in Fig. 1.1*a* varies with time, it would be necessary to know the displacement and velocity of all points in the spring at some specific instant of time in order to be able to predict the transient deflections within the spring at all subsequent times. In addition, it would be necessary to know all the forces acting on the boundary of the spring. A static field problem is called a "boundary-value problem." Since the excitations and responses are all constant in time, a specification of their magnitude and location within the field is sufficient for the solution of the analysis problem. Both these types of analysis problems have unique solutions, that is, they have one and only one correct solution. Given the excitation and the system, the response is completely and unambiguously determined.

At this point, it is important to make a distinction between scientific and engineering analysis. The pure scientist is impelled to find complete descriptions for physical phenomena. In the case of a field problem, he considers the problem solved only if he has determined the exact system behavior at all points in the field for all time. When an engineer attacks a field problem, on the other hand, he always has a specific objective in mind (which can probably eventually be translated into dollars and

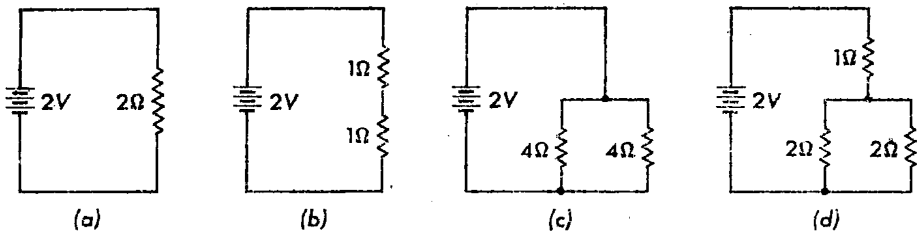

FIG. 1.3. Some possible ways of synthesizing a 2-ohm impedance.

cents). The engineer may ask, in the case of a heat-transfer problem: "Where in the thermal conductor is the temperature a maximum? Where is the thermal gradient the largest? What is the temperature along a certain boundary? How long does it take for the conductor to cool to a certain temperature? etc." He does not ask: "What is the temperature in the conductor at all points and at all times?" The aim of engineering analysis is not mathematical description. It is, rather, to obtain specific answers within a specified accuracy at a minimum cost in time, labor, and equipment.

The synthesis problem does not have a unique solution. In engineering *synthesis* or design, the excitation is specified and a specified response is to be obtained. The problem is to find or design a system that will yield this response. The nonuniqueness of the synthesis problem is easily demonstrated by referring to electric-circuit theory. Suppose that it is desired to synthesize a network which will draw a current of 1 amp (response) from a 2-volt source (excitation). A few possible circuit configurations, representing solutions to this simple synthesis problem, are indicated in Fig. 1.3. Evidently there are an infinite number of solutions. The questions of optimization therefore arise. Perhaps a minimum number of elements are desirable; in that case the circuit shown in Fig. 1.3*a* is best. Perhaps it is preferred that the current through each element not exceed ½ amp or that the voltage drop across the element not exceed 1 volt. In each case, a different circuit represents an optimum solution. The same consideration evidently applies to field problems.