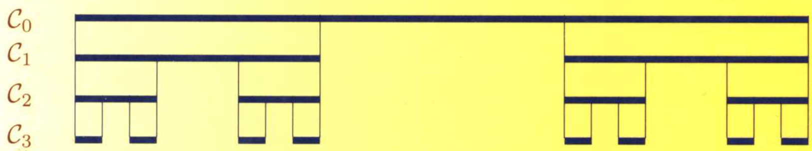
Yiannis Moschovakis

NOTES ON SET THEORY

Second Edition



Springer

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Notes on Set Theory

Second Edition

With 48 Figures

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(continued after index)

Dedicated to the memory of Nikos Kritikos

PREFACE

What this book is about. The *theory of sets* is a vibrant, exciting mathematical theory, with its own basic notions, fundamental results and deep open problems, and with significant applications to other mathematical theories. At the same time, *axiomatic set theory* is often viewed as a *foundation of mathematics*: it is alleged that all mathematical objects are sets, and their properties can be derived from the relatively few and elegant axioms about sets. Nothing so simple-minded can be quite true, but there is little doubt that in standard, current mathematical practice, "making a notion precise" is essentially synonymous with "defining it in set theory". Set theory is the official language of mathematics, just as mathematics is the official language of science.

Like most authors of elementary, introductory books about sets, I have tried to do justice to both aspects of the subject.

From straight set theory, these Notes cover the basic facts about "abstract sets", including the Axiom of Choice, transfinite recursion, and cardinal and ordinal numbers. Somewhat less common is the inclusion of a chapter on "pointsets" which focuses on results of interest to analysts and introduces the reader to the Continuum Problem, central to set theory from the very beginning. There is also some novelty in the approach to cardinal numbers, which are brought in very early (following Cantor, but somewhat deviously), so that the basic formulas of cardinal arithmetic can be taught as quickly as possible. Appendix **A** gives a more detailed "construction" of the real numbers than is common nowadays, which in addition claims some novelty of approach and detail. Appendix **B** is a somewhat eccentric, mathematical introduction to the study of *natural models* of various set theoretic principles, including Aczel's Antifoundation. It assumes no knowledge of logic, but should drive the serious reader to study it.

About set theory as a foundation of mathematics, there are two aspects of these Notes which are somewhat uncommon. First, I have taken seriously this business about "everything being a set" (which of course it is not) and have tried to make sense of it in terms of the notion of *faithful representation* of mathematical objects by *structured sets*. An old idea, but perhaps this is the first textbook which takes it seriously, tries to explain it, and applies it consistently. Those who favor category theory will recognize some of its basic notions in places, shamelessly folded into a traditional set theoretical approach to the foundations where categories are never mentioned. Second, *computation theory* is viewed as part of the mathematics "to be founded" and the relevant set theoretic results have been included, along with several examples. The ambition was to explain what every young mathematician or theoretical computer scientist needs to know about sets.

The book includes several historical remarks and quotations which in some places give it an undeserved scholarly gloss. All the quotations (and most of the comments) are from papers reprinted in the following two, marvellous and easily accessible source books, which should be perused by all students of set theory:

Georg Cantor, *Contributions to the founding of the theory of transfinite numbers*, translated and with an Introduction by Philip E. B. Jourdain, Dover Publications, New York.

Jean van Heijenoort, *From Frege to Gödel*, Harvard University Press, Cambridge, 1967.

How to use it. About half of this book can be covered in a Quarter (ten weeks), somewhat more in a longer Semester. Chapters **1** – **6** cover the beginnings of the subject and they are written in a leisurely manner, so that the serious student can read through them alone, with little help. The trick to using the Notes successfully in a class is to cover these beginnings very quickly: skip the introductory Chapter **1**, which mostly sets notation; spend about a week on Chapter **2**, which explains Cantor's basic ideas; and then proceed with all deliberate speed through Chapters **3** – **6**, so that the theory of well ordered sets in Chapter **7** can be reached no later than the sixth week, preferably the fifth. Beginning with Chapter **7**, the results are harder and the presentation is more compact. How much of the "real" set theory in Chapters **7** – **12** can be covered depends, of course, on the students, the length of the course, and what is passed over. If the class is populated by future computer scientists, for example, then Chapter **6** on Fixed Points should be covered in full, with its problems, but Chapter **10** on Baire Space might be omitted, sad as that sounds. For budding young analysts, at the other extreme, Chapter **6** can be cut off after **6.27** (and this too is sad), but at least part of Chapter **10** should be attempted. Additional material which can be left out, if time is short, includes the detailed development of addition and multiplication on the natural numbers in Chapter **5**, and some of the less central applications of the Axiom of Choice in Chapter **9**. The Appendices are quite unlikely to be taught in a course (I devote just one lecture to explain the idea of the construction of the reals in Appendix **A**), though I would like to think that they might be suitable for undergraduate Honors Seminars, or individual reading courses.

Since elementary courses in set theory are not offered regularly and they are seldom long enough to cover all the basics, I have tried to make these Notes accessible to the serious student who is studying the subject on their own. There are numerous, simple Exercises strewn throughout the text, which test understanding of new notions immediately after they are introduced. In class I present about half of them, as examples, and I assign some of the rest for easy homework. The Problems at the end of each chapter vary widely in difficulty, some of them covering additional material. The hardest problems are marked with an asterisk (*).

Acknowledgments. I am grateful to the Mathematics Department of the University of Athens for the opportunity to teach there in Fall 1990, when I wrote the first draft of these Notes, and especially to Prof. A. Tsarpalias who usually teaches that Set Theory course and used a second draft in Fall 1991; and to Dimitra Kitsiou and Stratos Paschos for struggling with PCs and laser printers at the Athens Polytechnic in 1990 to produce the first "hard copy" version. I am grateful to my friends and colleagues at UCLA and Caltech (hotbeds of activity in set theory) from whom I have absorbed what I know of the subject, over many years of interaction. I am especially grateful to my wife Joan Moschovakis and my student Darren Kessner for reading large parts of the preliminary edition, doing the problems and discovering a host of errors; and to Larry Moss who taught out of the preliminary edition in the Spring Term of 1993, found the remaining host of errors and wrote out solutions to many of the problems.

The book was written more-or-less simultaneously in Greek and English, by the magic of bilingual LaTeXand in true reflection of my life. I have dedicated it to Prof. Nikos Kritikos (a student of Caratheodory), in fond memory of many unforgettable hours he spent with me back in 1973, patiently teaching me how to speak and write mathematics in my native tongue, but also much about the love of science and the nature of scholarship. In this connection, I am also greatly indebted to Takis Koufopoulos, who read critically the preliminary Greek version, corrected a host of errors and made numerous suggestions which (I believe) improved substantially the language of the final Greek draft.

Palaion Phaliron, Greece November 1993

About the 2nd edition. Perhaps the most important changes I have made are in small things, which (I hope) will make it easier to teach and learn from this book: simplifying proofs, streamlining notation and terminology, adding a few diagrams, rephrasing results (especially those justifying *definition by recursion*) to ease their applications, and, most significantly, correcting errors, typographical and other. For spotting these errors and making numerous, useful suggestions over the years, I am grateful to Serge Bozon, Joel Hamkins, Peter Hinman, Aki Kanamori, Joan Moschovakis, Larry Moss, Thanassis Tsarpalias and many, many students.

The more substantial changes include:

— A proof of *Suslin's Theorem* in Chapter **10**, which has also been significantly massaged.

— A better exposition of ordinal theory in Chapter **12** and the addition of some material, including the basic facts about ordinal arithmetic.

— The last chapter, a compilation of solutions to the Exercises in the main part of the book – in response to popular demand. This eliminates the most obvious, easy homework assignments, and so I have added some easy problems.

I am grateful to Thanos Tsouanas, who copy-edited the manuscript and caught the worst of my mistakes.

Palaion Phaliron, Greece July 2005

CONTENTS

CHAPTER 1

INTRODUCTION

Mathematicians have always used sets, e.g., the ancient Greek geometers defined a circle as the set of points at a fixed distance r from a fixed point C, its center. But the systematic study of sets began only at the end of the 19th century with the work of the great German mathematician Georg Cantor, who created a rigorous theory of the concept of *completed infinite* by which we can compare infinite sets as to size. For example, let

$$\mathbb{N} = \{0, 1, \dots\} = \text{the set of natural numbers,}$$
$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\} = \text{the set of rational integers,}$$
$$\mathbb{Q} = \text{the set of rational numbers (fractions),}$$
$$\mathbb{R} = \text{the points of a straight line,}$$

where we also identify $\mathbb{R}$ with the set of real numbers, each point associated with its (positive or negative) coordinate with respect to a fixed origin and direction. Cantor asked if these four sets "have the same (infinite) number of elements", or if one of them is "more numerous" than the others. Before we make precise and answer this question in the next chapter, we review here some basic, well-known facts about sets and functions, primarily to explain the notation we will be using.

What are sets, anyway? The question is like "what are points", which Euclid answered with

a point is that which has no parts.

This is not a rigorous mathematical definition, a reduction of the concept of "point" to other concepts which we already understand, but just an intuitive description which suggests that a point is some thing which has no extension in space. Like that of point, the concept of set is fundamental and cannot be reduced to other, simpler concepts. Cantor described it as follows:

> By a set we are to understand any collection into a whole of definite and separate objects of our intuition or our thought.

Vague as it is, this description implies two basic properties of sets.

1. Every set A has **elements** or **members**. We write

$$x \in A \iff \text{the object } x \text{ is a member of (or belongs to) } A.$$

2. A set is determined by its members, i.e., if A, B are sets, then[1]

$$A = B \iff A \text{ and } B \text{ have the same members} \qquad (1\text{-}1)$$
$$\iff (\forall x)[x \in A \iff x \in B].$$

This last is the **Extensionality Property**. For example, the set of students in this class will not change if we all switch places, lie down or move to another classroom; this set is completely determined by *who we are*, not our posture or the places where we happen to be.

Somewhat peculiar is the **empty set** $\emptyset$ which has no members. The extensionality property implies that *there is only one empty set.*

If A and B are sets, we write

$$A \subseteq B \iff (\forall x)[x \in A \Longrightarrow x \in B],$$

and if $A \subseteq B$, we call A a **subset** of B, so that for every B,

$$\emptyset \subseteq B, \quad B \subseteq B.$$

A **proper subset** of B is a subset distinct from B,

$$A \subsetneq B \iff [A \subseteq B \,\&\, A \neq B].$$

From the extensionality property it follows that for all sets A, B,

$$A = B \iff A \subseteq B \,\&\, B \subseteq A.$$

We have already used several different notations to define specific sets and we need still more, e.g.,

$$A = \{a_1, a_2, \dots, a_n\}$$

is the (finite) set with members the objects $a_1, a_2, \dots, a_n$. If P is a condition which specifies some property of objects, then

$$A = \{x \mid P(x)\}$$

is the set of all objects which satisfy the condition P, so that for all x,

$$x \in A \iff P(x).$$

For example, if

$$P(x) \iff x \in \mathbb{N} \,\&\, x \text{ is even},$$

then $\{x \mid P(x)\}$ is the set of all even. natural numbers. We use a variant of this notation when we are only interested in "collecting into a whole" members of a given set A which satisfy a certain condition:

$$\{x \in A \mid P(x)\} =_{\text{df}} \{x \mid x \in A \,\&\, P(x)\},$$

[1] We will use systematically, as abbreviations, the logical symbols

$\&$: and, $\vee$: or, $\neg$: not, $\Longrightarrow$: implies, $\iff$: if and only if,

$\forall$: for all, $\exists$: there exists, $\exists!$: there exists exactly one.

The symbols $=_{\text{df}}$ and $\iff_{\text{df}}$ are read "equal by definition" and "equivalent by definition".

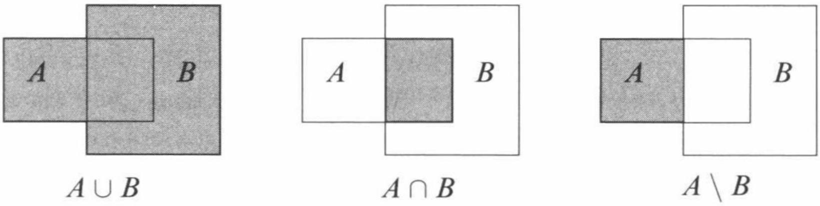

FIGURE **1.1**. The Boolean operations.

so that, for example, $\{x \in \mathbb{N} \mid x > 0\}$ is the set of all non-zero natural numbers, while $\{x \in \mathbb{R} \mid x > 0\}$ is the set of all positive real numbers.

For any two sets[2] A, B,

$$\begin{aligned} A \cup B &= \{x \mid x \in A \vee x \in B\} && \text{(the union of } A, B\text{)},\\ A \cap B &= \{x \in A \mid x \in B\} && \text{(the intersection of } A, B\text{)},\\ A \setminus B &= \{x \in A \mid x \notin B\} && \text{(the difference of } A, B\text{)}. \end{aligned}$$

These "Boolean operations" are illustrated in the so-called *Venn diagrams* of Figure **1.1**, in which sets are represented by regions in the plane. The union and the intersection of infinite sequences of sets are defined in the same way,

$$\bigcup_{n=0}^{\infty} A_n = A_0 \cup A_1 \cup \cdots = \{x \mid (\exists n \in \mathbb{N})[x \in A_n]\},$$

$$\bigcap_{n=0}^{\infty} A_n = A_0 \cap A_1 \cap \cdots = \{x \mid (\forall n \in \mathbb{N})[x \in A_n]\}.$$

Two sets are **disjoint** if their intersection is empty,

$$A \text{ is disjoint from } B \iff A \cap B = \emptyset.$$

We will use the notations

$$f : X \to Y \text{ or } A \xrightarrow{f} B$$

to indicate that f is a **function** which associates with each member x of the set X, the **domain** of f some member $f(x)$ of the **range** Y of f. Functions are also called **mappings**, **operations**, **transformations** and many other things. Sometimes it is convenient to use the abbreviated notation $(x \mapsto f(x))$ which makes it possible to talk about a function without officially naming it. For example,

$$(x \mapsto x^2 + 1)$$

is the function on the real numbers which assigns to each real its square increased by 1; if we call it f, then it is defined by the formula

$$f(x) = x^2 + 1 \quad (x \in \mathbb{R})$$

[2]In "mathematical English", when we say "for any two objects x, y", we do not mean that necessarily $x \neq y$, e.g., the assertion that "for any two numbers x, y, $(x+y)^2 = x^2 + 2xy + y^2$" implies that "for every number x, $(x+x)^2 = x^2 + 2xx + x^2$".

so that $f(0) = 1$, $f(2) = 5$, etc. But we can say "all the values of $(x \mapsto x^2+1)$ are positive reals" without necessarily fixing a name for it, like f.

Two functions are **equal** if they have the same domain and they assign the same value to every member of their common domain,

$$f = g \iff (\forall x \in X)[f(x) = g(x)] \quad (f : X \to Y, g : X \to Z, x \in X).$$

In connection with functions we will also use the notations

$$\begin{aligned} f : X \rightarrowtail Y &\iff_{\text{df}} f \text{ is an } \textbf{injection} \text{ (one-to-one)} \\ &\iff (\forall x, x' \in X)[f(x) = f(x') \Longrightarrow x = x'], \\ f : X \twoheadrightarrow Y &\iff_{\text{df}} f \text{ is a } \textbf{surjection} \text{ (onto)} \\ &\iff (\forall y \in Y)(\exists x \in X)[f(x) = y], \\ f : X \rightarrowtail\!\!\!\!\twoheadrightarrow Y &\iff_{\text{df}} f \text{ is a } \textbf{bijection} \text{ or a } \textbf{correspondence} \\ &\iff (\forall y \in Y)(\exists! x \in X)[f(x) = y]. \end{aligned}$$

For every $f : X \to Y$ and $A \subseteq X$, the set

$$f[A] =_{\text{df}} \{f(x) \mid x \in A\}$$

is the **image** of A under f, and if $B \subseteq Y$, then

$$f^{-1}[B] =_{\text{df}} \{x \in X \mid f(x) \in B\}$$

is the **pre-image** of B by f.

If f is a bijection, then we can define the **inverse function** $f^{-1} : Y \to X$ by the condition

$$f^{-1}(y) = x \iff f(x) = y,$$

and then the inverse image $f^{-1}[B]$ (as we defined it above) is precisely the image of B under f^{-1}.

The **composition**

$$h =_{\text{df}} gf : X \to Z$$

of two functions

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is defined by

$$h(x) = g(f(x)) \quad (x \in X).$$

It is easy to prove many basic properties of sets and functions using only these definitions and the extensionality property. For example,

$$A \cup B = B \cup A,$$

because, for any x,

$$\begin{aligned} x \in A \cup B &\iff x \in A \text{ or } x \in B \\ &\iff x \in B \text{ or } x \in A \\ &\iff x \in B \cup A. \end{aligned}$$

In some cases, the logic of the argument gets a bit complex and it is easier to prove an identity $U = V$ by verifying separately the two implications

$x \in U \Longrightarrow x \in V$ and $x \in V \Longrightarrow x \in U$. For example, if $f : X \to Y$ and $A, B \subseteq X$, then

$$f[A \cup B] = f[A] \cup f[B].$$

To prove this, we show first that

$$x \in f[A \cup B] \Longrightarrow x \in f[A] \cup f[B];$$

this holds because if $x \in f[A \cup B]$, then there is some $y \in A \cup B$ such that $x = f(y)$; and if $y \in A$, then $x = f(y) \in f[A] \subseteq f[A] \cup f[B]$, while if $y \in B$, then $x = f(y) \in f[B] \subseteq f[A] \cup f[B]$. Next we show the converse implication, that

$$x \in f[A] \cup f[B] \Longrightarrow x \in f[A \cup B];$$

this holds because if $x \in f[A]$, then $x = f(y)$ for some $y \in A \subseteq A \cup B$, and so $x \in f[A \cup B]$, while if $x \in f[B]$, then $x = f(y)$ for some $y \in B \subseteq A \cup B$, and so, again, $x \in f[A \cup B]$.

Problems for Chapter 1

x1.1. For any three sets A, B, C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$
$$A \setminus (A \cap B) = A \setminus B.$$

x1.2. (De Morgan's laws) For any three sets A, B, C,

$$C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B),$$
$$C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B).$$

x1.3. (De Morgan's laws for sequences) For any set C and any sequence of sets $\{A_n\}_n = A_0, A_1, \ldots,$

$$C \setminus (\bigcup_n A_n) = \bigcap_n (C \setminus A_n),$$
$$C \setminus (\bigcap_n A_n) = \bigcup_n (C \setminus A_n).$$

x1.4. For every injection $f : X \rightarrowtail Y$, and all $A, B \subseteq X$,

$$f[A \cap B] = f[A] \cap f[B],$$
$$f[A \setminus B] = f[A] \setminus f[B].$$

Show also that these identities do not always hold if f is not an injection.

x1.5. For every $f : X \to Y$, and all $A, B \subseteq Y$,

$$f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B],$$
$$f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B],$$
$$f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B].$$