ELEMENTS OF SET THEORY

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ALLYN AND BACON, INC. • BOSTON

PREFACE

Our justification for a separate course in set theory, which we were called upon to organize and teach, for upper division undergraduate and beginning graduate students at Colorado State College, was based on reports of various committees devoted to the study of the undergraduate curriculum, on our own experiences in learning and teaching mathematics, and on information gained from our meetings, both formal and informal, with high school teachers of mathematics. The course was to meet three hours per week for one quarter and would consist of students who had some degree of acquaintance with the real number system and elementary real functions. For some, the course would be a prerequisite to other required courses; for others, it would be a terminal course in their program; and for both groups, it would very likely be their first exposure to modern mathematics. Also, many of our students would be high school teachers who, faced with the problem of teaching some aspects of set theory in their own classes, were returning to the campus to extend their knowledge of the subject matter.

Although various courses in modern mathematics can be, and are, used to meet the above requirements, none, we felt, could be more basic than set theory, which occurs in one form or another in all modern subject matter in mathematics. A course designed to meet the needs of such a varied group of students as we have described must contain certain obvious ingredients. If it is to serve as a prerequisite to other courses, it should provide specific tools and techniques which the student may have occasion to employ in later applications. In this regard, special emphasis on the concept of function and related topics seemed appropriate. To deserve the name modern mathematics, the course should make the reader aware of the fact that modern mathematics does not merely say old things in a new way; we felt that the concept of transfinite cardinal numbers is an excellent means (though not the only one, of course) of illustrating this point. Finally, set theory, as a central branch of mathematics with its own assumptions, is an ideal medium in which axiomatic structure and the nature of proof can be studied.

These, and other, thoughts prompted us to adopt, as our central theme for the course, the systematic development of cardinal numbers and their arithmetic in such a way that the basic tools of set theory would be developed with emphasis placed on the subject matter as a mathematical system with a logical structure. Clearly, our course would have to be more than an advanced treatment of Venn diagrams. At the same time, the background and level of our students imposed obvious limitations on the degree to which axiomatic structure could be utilized. What was needed was a semi-axiomatic treatment in which attention to assumptions and rigor is emphasized but not to the extent of seriously hampering student progress or, worse, producing a negative attitude toward modern mathematics.

A search for a suitable textbook for the course revealed that those available were generally of two types. On the one hand, there were the rather complete axiomatic treatments which required a level of sophistication beyond that of our course. At the other extreme were to be found a few materials at the introductory level, most of which were entirely too brief or elementary for our purposes. Therefore, encouraged by our students, we began duplicating our own teaching notes to fill this apparent gap. These notes evolved into the present book.

Our treatment is axiomatic but is by no means complete. We use just those axioms which we found, after class trial, to be within the grasp of our students and yet sufficient to achieve our purpose with a certain degree of rigor. When rigor has been sacrificed we have tried to be honest with the reader by pointing it out. Anyone who attempts to strike such a balance between intuition and rigor runs the risk of including too much of some things and not enough of others. We can say of the materials presented here that they have been used with varying degrees of success in several classes, some of which consisted almost entirely of high school teachers. Reactions from those students as they continue their study and teaching have been encouraging.

We make no apology for the brief, and perhaps inadequate, treatment of logic in Chapter 1. In the early versions of the manuscript it was omitted altogether. We soon found, however, that it was necessary to teach some material of this type, and we have included just that amount which we found needed in the remaining chapters. No serious attempt is made in Chapter 1 to illustrate the logical notions developed, since we are relying on the ensuing material to illustrate these ideas extensively.

After introducing the notions of sets and their operations in Chapter 2, careful attention is given to the matter of proof by means of basic theorems about sets. Functions and related topics are treated extensively in Chapter 3. As sets, they illustrate the material developed in Chapter 2 and are indispensable for the discussion of equivalence of sets which follows in Chapter 4. The pace through these chapters is necessarily slow. At times, painstaking effort is devoted to proofs, and we include details that are often left to the reader. Such slow progress, however, is not without its reward, and we have found that beginning students particularly appreciate having detailed proofs at hand to serve as models when they are called upon to construct proofs for themselves.

In Chapter 5, cardinal numbers are introduced only after fixing the notion of equivalence of sets in Chapter 4. The arithmetic of cardinal numbers is given and coordinates the efforts of all the preceding chapters. As the reader progresses through these chapters, he will find the amount of symbolism and abstractness steadily increasing. There are at least two reasons for this. In the first place, much of the power and elegance of mathematics in general, and set theory in particular, lies in its symbolism and abstractness. To present mathematics in any other way is not fair to the student and may, in fact, even create a false impression of what constitutes modern mathematics. Secondly, although we seriously attempt to maintain a somewhat elementary level, we feel compelled to raise the student's level as he progresses through his study of the material. Recognizing the risk of defeating our own purpose by including too much of the abstract, we have tried to motivate and discuss each definition and the main results before presenting them.

With the main purpose of the book accomplished in Chapter 5, we turn to a brief discussion of ordered sets and other topics in Chapter 6. The length and scope of the course prevent us from going into great detail in these matters, but we hope the treatment is adequate enough to develop insights and arouse sufficient curiosity to prompt further study on the part of the reader.

In making acknowledgments, we would like to thank first our students who suffered through the early manuscripts. High school teachers in the various classes were especially helpful in assisting us in gauging the level of treatment. We would also like to thank the Science Division of Colorado State College for assistance in providing secretarial services. Special mention should be made of Miss Joyce Ridgel and Mrs. Glen Turner for their help in typing the final manuscript. Finally, we must thank our publishers and their reviewers for their invaluable suggestions and encouragement.

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A FOREWORD TO THE READER

The kind of mathematics included in this text may be quite different from any you have previously studied. If this is the case, then the following suggestions on how to study this material may be of help, as they have been to our own students.

We strongly urge that you memorize each definition as soon as it appears. We are not minimizing the importance of understanding, and in connection with each definition there will be examples and discussions to aid you in that understanding, but, since each topic is so dependent upon preceding ones, we have found that memorization of the definitions will speed your progress.

As each new theorem is stated, stop reading and make a real attempt to prove the theorem by yourself. You may not be successful in every case (especially in the early topics), but the attempt will aid in pointing out the importance of previous material. Then, after reading the proof given in the text, make sure that the method of proof we have used is clear. If our proof differs from your own, either show that yours is an equivalent proof or correct it.

The exercises given at the end of most of the sections are designed not only to give practice in the use of the definitions and theorems but also to give you the opportunity to do some independent thinking. Be sure to work each of these exercises carefully.

Finally, when the study of a chapter is complete, make a summary of definitions, axioms, and theorems included in that chapter and test your understanding of each by sketching proofs and giving examples. We have provided a glossary of symbols on page 173 to assist you in summarizing material and to provide for easy reference.

TO OUR WIVES, VERONICA AND SALLIE

 • Library of Congress Catalog Card Number: 62-14758 • Printed in the United States of America

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INTRODUCTION AND ELEMENTARY LOGIC

1.1 HISTORICAL REMARKS

The notion of set or collection is probably as primitive as the notion of number. In fact, as we shall see, the two are not entirely unrelated. For example, it has been said that when a child hears the word "two" he thinks of a set consisting of two objects of his experience. Hence, the idea of collecting certain objects into a single whole seems to be quite natural. To the mathematician of the twentieth century the dividends which accrue to mathematics from this simple idea are many, and one might have expected some discoveries in this area centuries ago. However, it was not until the latter part of the nineteenth century that the German mathematician Georg Cantor (1845–1918) proposed the first formal treatment of sets as mathematical entities.

In his research in analysis Cantor arrived at a point where it seemed necessary to generalize the concept of number beyond the usual finite sense. He needed a concept that would introduce an actual infinity into mathematics.* As Cantor proceeded in this research he discovered that he could not only formalize the notion of infinite number but also classify various types of infinite numbers! To see how bold this gesture was in Cantor's contemporary world of mathematics, one has only to realize that no less a mathematician than Frederick Gauss (in 1831) had rejected the idea of an actual infinite, considering such an idea to be inadmissible in mathematics. Moreover, the early Greeks, and most mathematicians since their time, had felt that the concept of infinity was germane to the study of theology and philosophy, not mathematics.†

It is true that an "infinity" of sorts existed in mathematics before Cantor's work. However, this "infinity" came from a consideration of limits, the study of which was one of the central themes of the research in mathematics during the eighteenth and nineteenth centuries. For example, the expression $1/x^2$ was said to become large beyond finite bound as x was assigned smaller and smaller values. Indeed, the mathematical shorthand we use today, namely, "$\lim_{x\to 0} 1/x^2 = \infty$," reflects the acceptance of this version of an "infinity." But it was never intended that this concept introduced an actual infinite. On the contrary, the symbolism and the various phrases used, such as "$1/x^2$ becomes infinite as x approaches zero," were intended to abbreviate the precise idea that given any positive number k, it is possible to find a positive number h such that $1/x^2 > k$ whenever $0 < |x| < h$.

Granted that an actual infinite number had not been intended through the use of limits, why not admit into our number system a number which might be called "infinity" and which would have properties that might be expected of it in terms of the corresponding

* We are using the words "set," "finite," and "infinite" in a purely informal manner in this introduction, and we ask the reader to rely upon his intuitive understanding of these words at this point.

† Abraham A. Fraenkel, *Abstract Set Theory* (Amsterdam: Holland Publishing Company, 1961), p. 1.

limit relations? One of the answers is that if we do so, we lose some of the valuable properties of our operations in the number system. For example, one natural property of this number "infinity" (dare we write ∞?) is that given any real number a we should have $a + \infty = \infty$. However, one of the important properties of addition in the real number system is the cancellation law. That is, if a, b, and c are real numbers and if $a + c = b + c$ then $a = b$. Notice that we lose this cancellation law if we include ∞ as a real number, for if we let $a = 1$, $b = 2$ and $c = \infty$, the cancellation law, if true, would show that $1 = 2$, a fact we cannot accept.

In the light of these remarks it is not too surprising that Cantor's works were not generally accepted by his contemporaries. However, as his work progressed and as other mathematicians, who had found wide application of the theory, also joined in the work, the importance of the subject became more apparent. Subsequently, this field of study was labeled "set theory."

Ironically, about the time that Cantor's set theory began to gain acceptance, certain inconsistencies, called paradoxes, were discovered in the higher reaches of the subject. One might think that because of the early reluctance to accept set theory these discoveries would have sounded the death knell for the subject. However, interest in the theory had reached such a peak that other mathematicians such as Hilbert, Fraenkel, and Zermelo began a serious investigation of the reasons for these paradoxes and made several outstanding discoveries in their attempts to resolve them. In the sixty-four years since the appearance of the first of the paradoxes, by Burali-Forti in 1897, progress has been made in resolving these issues, but at the date of this writing some are still without resolution.

1.2 THE ROLE OF SET THEORY IN MATHEMATICS

The growth of mathematics in the period from about 1900 to the present has been outstanding to say the least. Set theory has played no minor role in the development of what we call modern mathematics and, as a matter of fact, may be said to be the basis for several of its branches. Even the paradoxes of set theory helped in the development of an area of mathematics known as foundations, wherein the axiomatic structure of all mathematics is crucially investigated. From a more direct point of view we now consider nearly every branch of mathematics to be a study of sets of objects of one kind or another. Thus, geometry is a study of sets of points; algebra deals with sets of numbers (or prototypes thereof) and operations on those sets. Analysis is mainly concerned with functions and the latter, as we shall see, are merely sets of a particular kind. We recognize the theory of sets as a unifier and simplifier in present-day mathematics. It is a unifier in that the language and properties of sets are extensively used in diverse branches of mathematics; it serves as a simplifier in that sets, by their very nature, treat collections of objects as entities in themselves and hence provide a convenient notation for handling those entities.

On the other hand, set theory may be viewed as a branch of mathematics having its own particular assumptions and structure. As such it deserves study for its own sake. In fact, in answer to the question which is so often posed by the student as to why set theory should be studied, we would say that first and foremost it is mathematics. If the reader has not already encountered a need for and use of set theory in his study of mathematics, it is unlikely that he will proceed much further without meeting that need and use. The main purpose of this book is to relate to the reader how Cantor's generalization of number, considered by many to be the most remarkable achievement of modern mathematics, can be accomplished through a study of sets and their properties.

To accomplish our avowed purpose, it will be necessary first to examine the structure of sets as mathematical objects. In this regard, there are essentially two avenues of approach. We mentioned previously that certain paradoxes in set theory arose shortly after Cantor's works were published. Some of these came about because of the highly intuitive nature of Cantor's definitions and