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ISBN 978-3-319-38944-8 ISBN 978-3-319-38945-5 (eBook)
DOI 10.1007/978-3-319-38945-5

Library of Congress Control Number: 2016939938

Mathematics Subject Classification (2010): 49-03, 49-01, 01A45, 01A50, 01A75

Printed on acid-free paper

This book is published under the trade name Birkhäuser
The registered company is Springer International Publishing AG Switzerland
(www.birkhauser-science.com)

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We dedicate this book
to our respective grandchildren
Michela, Gabriele and Andrea

Preface

> In the current state of analysis we may regard these discussions [of past mathematics] as useless, for they concern forgotten methods, which have given way to others more simple and more general. However, such discussions may yet retain some interest for those who like to follow step by step the progress of analysis, and to see how simple and general methods are born from particular questions and complicated and indirect procedures.[1]
>
> J.L. Lagrange, *Leçons sur le calcul des fonctions*, Paris 1806, p. 436.

The early history of the Calculus of variations is a well-beaten track; for instance, we refer the reader to

- The last two chapters of the *Calcul des functions* of Lagrange [152];
- *A Treatise on Isoperimetrical Problems and the Calculus of Variations* by R. Woodhouse, reprinted by Chelsea with the title *A History of the Calculus of Variations in the Eighteenth century*, [202];
- The surveys by C. Carathéodory

  (1) *The beginning of research in the Calculus of variations*, [48],
  (2) *Basel und der Begin der Variationsrechnung*, [49],
  (3) *Einführung in Eulers Arbeiten über Variationsrechnung*, [50],
  and the two volumes

  (1) *Variationsrechnung*, [46],
  (2) *Geometrische Optik*, [47];

[1]Quoted in [102], the original being
Mais dans l'état actuel de l'analyse, on peut regarder ces discussions comme inutiles, parceque elles regardent des méthodes oubliées, comme ayant fait place à d'autres plus simples et plus générales. Cependant elle peuvent avoir encore quelqu'intérêt pour ceux qui aiment suivre pas à pas les progrès de l'analyse, et à voir comment les méthodes simple et générales naissent des questions particulières et des procédés indirects et compliqués.

- the very detailed survey of the history of the one-dimensional calculus of variations from the origin until the beginning of last century by Goldstine [117], and by Thiele [193], which includes also some multidimensional calculus;
- the papers by Fraser [102, 104, 105]

Finally, we mention the two introductions to classical calculus of variations [33] and [112] which contain some historical references.

Nevertheless, we would like to go over once again presenting the most relevant continental contributions to the calculus of variations in the eighteenth century. Our main goal is to illustrate the mathematics of its founders. In doing this, we always follow very closely the original papers in their mathematical context and often in an almost literal way, adding, when we feel it is useful, our mathematical comment or complementing their proofs; however, we keep our additions separate from the original presentation. Here and there, we also comment in terms of modern mathematics. In fact, we think that this may help the reader to make clearer what ancient authors were doing, the difficulties they had to face, mistakes they made and how they were able to handle the matter following their approaches. We added the final Sect. 7.6 to make the reader, not necessarily an expert, aware of the end of the story, that is, of how the entire material is treated today.

Our book is addressed not only to historians of mathematics, but also to mathematicians who want to follow "step by step the progress of analysis" and to students of mathematics who, this way, may see the forming of a beautiful theory and the evolving of mathematical methods and techniques. This way, we hope that our work may help in getting a better understanding of the mathematical results, of the methods and techniques to obtain them, as well as of the mathematical historical context in which it all developed. Of course, in doing that, we take advantage of the wide literature that we have partly already mentioned and to which we would like to acknowledge our gratitude.

We now shortly outline the content of each chapter. We begin with an introductory chapter where, after stating Johann Bernoulli's challenge that marks the beginning of the calculus of variations, we briefly illustrate issues that belong to periods before the challenge and are especially relevant for our story: Fermat's principle of least action, which plays a crucial role in solving the brachistochrone problem; how previous minimum problems, as for instance the classical isoperimetric problem, differ from the problem of least time descent; what Johann and Jakob Bernoulli, Leonhard Euler and Joseph Louis Lagrange meant for solutions of the new minimum problems. Since most of the beginning of the calculus of variations is based on the notion of "infinitesimal elements", in Sect. 1.3 we discuss briefly the notion of "differential" in Leibniz and Euler and, with the aim of clarifying some of the claims of the early papers of the Calculus of variations, we illustrate in Sect. 1.4 the geometrical and analytical treatment of the cycloid in the period.

Chapters 2–7 present a systematic, sufficiently complete and, we think, fair presentation of the works, actually of the mathematics in the relevant tracts of the Bernoullis, Euler and Lagrange, discussing also their connections, always being adherent to the original texts. In particular, Chap. 2 deals with the brachistochrone problem, Chap. 3 with the isoperimetric problem, according to the fundamental papers by Johann and Jakob Bernoulli. Chap. 4 deals with the beginning of the problem of finding geodesics on a surface with the contributions of Johann Bernoulli, Leonhard Euler and Alexis Clairaut. Chapters 5 and 6 deal with the key contributions of Euler to the isoperimetric problem, the former presenting the Memoirs of 1738 and 1741 that contain a famous error and the latter discussing the celebrated treatise *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentis*. Of course, we have no chance of discussing the many in specific minimum problems solved by Euler—surely one of the most beautiful and interesting aspects of the *Methodus inveniendi*—and we have to confine ourselves to discussing Euler's general method and illustrating only few examples. Finally, Chap. 7 presents the δ-calculus of Lagrange, first in the correspondence Lagrange–Euler and then in the main analytical treatise of Lagrange, adding a few more results of Lagrange that, however, belong more to the development of the calculus of variations in the nineteenth century. We conclude, in Sect. 7.5, with Euler's paper of 1771 that presents, we might say, the modern way of deriving the Euler–Lagrange equations expressing the necessary condition for minimality.

Topics in this volume were partially presented in a course–seminar held by the second author during the academic years 2011–2012 and 2012–2013 at the Scuola Normale Superiore in Pisa, dedicated to the development of calculus and mechanics in the cultural context of the eighteenth century. Expanded notes of these courses appeared as [111]. Special thanks go to friends, colleagues and students who actively participated contributing with relevant questions and very useful comments. We would like to thank particularly Vieri Benci, Sergio Bernini, Giuseppe Da Prato, Mauro Di Nasso, Marco Forti, Hykel Hosni and Massimo Mugnai. Also, we would like to thank Chiara Amadori, Federica D'Angelo, Daniela D'Innocenti and Andrea Tasini who prepared their master's theses on related topics under the supervision of the first author.

Finally, special warm thanks go to our friend Hykel Hosni who read the entire manuscript and helped to improve it with his comments and suggestions.

Last but not least, the second author would like to thank his friends Enrico Giusti, Stefan Hildebrandt, Giuseppe Modica and Jiři Souček whom he had the privilege to collaborate with for many years and who generously shared their ideas with him.

January 2016

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Contents

Chapter 1
Some Introductory Material

Minimum principles constitute one of the most beautiful and widespread paradigm in philosophy and the sciences. It is strongly related to the so-called principle of economy —what can be done, can be done simply[1] —and to the search for optimal strategies to realize our goals. This aesthetic and pragmatic concept also suggests the idea that nature proceeds in the simplest and most efficient way. As Newton wrote in his *Principia*:

> Nature does nothing in vain, and more is in vain when less will serve; for Nature is pleased with simplicity and affects not the pomp of superfluous causes.

Optimality principles have been used to formulate *laws of nature* since the very beginning of science, be it that such principles suit scientists aiming to unification and simplification of knowledge[2] or that they seem to reflect the preestablished harmony of our universe — Euler wrote in his *Methodus inveniendi* (the first treatise on calculus of variations):

> Because the shape of the whole universe is most perfect and, in fact, designed by the wisest creator, nothing in all of the world will occur in which no maximum or minimum rule is somehow shining forth.

---

[1]This is the *law of parsimony* often attributed to Ockham

> Entities are not to be multiplied beyond necessity;

in science it is often stated as

> What can be done with fewer assumptions is done in vain with more,

and was elevated to a virtue by Dante Alighieri, *De Monarchia*, Chapter XIV

> All that is superfluous displeases God and nature. All that displeases God and nature is evil.

[2]Max Born wrote in his *Physik im Wandel Meiner Zeit*

> It is science, not nature, to be economical.

P. Freguglia and M. Giaquinta, *The Early Period of the Calculus of Variations*, DOI 10.1007/978-3-319-38945-5_1

Little persists to date of Leibniz's belief in the best of all possible worlds and in the preestablished harmony of the universe; yet it remains the fact that many if not all laws of nature can be given the form of an extremal principle and many of the mathematical structures have their sources and their underlying texture in extremal principles.

The Calculus of variations — so named by Euler after the invention of the δ–calculus by Lagrange, replacing the old denomination *isoperimetric problems* — is a field of mathematics which deals with extremal problems, principles and methods to treat them. To be more detailed:

- It deals with specific minimum problems including, for instance, the study of geodesics on surfaces and of minimal surfaces both in codimension one or larger than one, of the gravitational potential and the Dirichlet principle (that Riemann put at the foundation of the theory of holomorphic functions), of the decomposition of harmonic differential forms (and the consequent study of the homology of a manifolds, a purely topological notion, in terms of differential forms), or of harmonic maps between manifolds, of constrained problems like in optimal control problems; in fact, specific problems are the key to general methods. In certain respects, one could say that the Calculus of variations is the art to finding optimal solutions and to describe their essential properties[3].
- The Lagrangian and Hamiltonian formalisms of the calculus that were developed in the Eighteenth and Nineteenth century turn out to be almost indistinguishable from the rational mechanics of systems of material points and offer a dual vision of mechanics and geometric optics; they eventually became the basis for the formulation of physical laws of nature (for continuum mechanics, electromagnetism and even modern fields theory or quantum mechanics, at least for stationary and conservative phenomena, and sometime even for nonconservative ones) and opened the way to modern symplectic geometry.
- The introduction of the so-called *direct methods* of the Calculus of variations for finding minimum points lead to modern functional analysis and geometric measure theory, and, in particular, to variational methods for the study of elliptic partial differential equations; while, the so-called global calculus of variations of Morse allowed to relate the topology of the space of competing functions with the number of critical points of energy functionals providing in particular existence of critical points in situations where for example no minimum point exists.

It goes without saying that this list could have been much more detailed in the topics mentioned, and overall a lot more comprehensive. Indeed after Newton and Leibniz invented calculus, the Calculus of variations grew more than exponentially both in quantity of relevant contributions and, qualitatively, in terms of providing an even deeper understanding of structures in mathematics and physics. This growth is still ongoing and the purpose of the above list of topics is solely that of giving the readers a feel of how vast the field is, while making them aware of how little of it we are going to cover.

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[3] As stated in the Introduction of [112].

In fact, in this volume, we shall deal only with some relevant issues developed in the first hundred years of life of Calculus of Variations, that is, in the Eighteenth century. Indeed, scholars mostly agree that June 1696 is the birth date of the Calculus of Variations.

1.1 Johann Bernoulli's Challenge

In June 1696 appeared in the *Acta Eruditorum*, as an appendix to the paper [19], *Problema novum ad cujus solutionem invitantur* [18] in which Johann Bernoulli challenged the 'geometers' to solve the following problem, which he would later call the *brachistochrone problem* and is also called the *problem of least time descent*:

> Given points *A* and *B* in a vertical plane to find the path *AMB* down which a movable point *M* must, by virtue of its weight, proceed from *A* to *B* in the shortest possible time (Figure 1.1).

To the description he added that the problem was relevant to mechanics, despite its appearence; and observed that its solution is not the straight line *AB*, but rather a curve which was *very well known* to geometers. In conclusion, Bernoulli announced that if no one had found the solution by the end of the year, he would have provided his own.

The problem was immediately solved by Leibniz[4] who also suggested to postpone the deadline to allow foreigners to receive the issue of *Acta Eruditorum* since its delivery outside Germany was apparently slow[5]. Johann Bernoulli agreed and in December 1696 announced (see [183] p. 646-648 for an English translation of the Groningen *Proclamation*) that the deadline had been extended to Easter 1967: If no one had succeeded by then in solving the problem, he would disclose Leibniz's and his own solution[6].

The May 1697 issue of the *Acta Eruditorum* appeared with Johann Bernoulli's solution on pp. 206-211, with the solution of his brother Jakob Bernoulli on pp. 211-218, with a brief note of presentation by Leibniz[7] saying that he would not

[4]Johann Bernoulli had posed his problem privately to him on 9 June 1696 and Leibniz's answer is dated 16 June, see Section 2.3.

[5]The events connected with the brachistochrone problem were reported by Johann Bernoulli in a letter to Henri Basnage sieur de Beauval (1657-1710), editor in Rotterdam from 1687 to 1709 of the *Histoire des Ouvrages des Savants* a kind of follower of the *Nouvelles de la République des Lettres* de Pierre Bayle (1647-1706); see [22] and [118] pp. 283-290.

[6]Westfall [200] claims that the challenge was for Newton

> Manifestly, both Bernoulli and Leibniz interpreted the silence from June to December as a demonstration that the problem had baffled Newton. They intended now to demonstrate their superiority publicly.

See Section 2.4 for more.

[7]*Comunicatio suae pariter duarum alienarum ad esendum sibi primum a dn. Joh. Bernoullio, deinde a dn. Marchionne Hospitalio communicatarum solutionum problematis curbae celerrimi descensus*

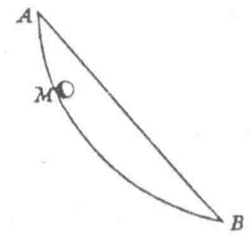

Fig. 1.1 The brachistochrone problem.

reproduce his solution since it was similar to that of the Bernoulli brothers[8], and with a discussion of the problem by l'Hôpital and Tschirnhaus.

However, the brachistochrone problem was not the first minimum problem in the history of mathematics, it was not even a new problem, as noticed by Leibniz: Most part of the third day of the *Discorsi e dimostrazioni matematiche intorno a due nuove scienze attinenti alla meccanica ed ai moti locali* by Galilei is dedicated to it[9]. Thus, the claim to the effect that the Calculus of variations begun in 1696 requires some motivation.

1.2 Before Johann Bernoulli

Let us first recall that if two sides of a triangle are different, then the angle opposite to the bigger side is larger than the angle opposed to the smaller side and, if two angles of a triangle are different, then the side opposed to the wider angle is longer than the side opposed to the smaller angle. It follows that in a triangle each side has length smaller than the sum of the lengths of the other two sides and larger than their difference, and that a necessary and also sufficient condition in order that x, y and z be the lengths of the sides of a triangle is that

$$x < y + z, \qquad y < x + z, \qquad z < x + y.$$

Another useful and immediate consequence of the above is

(Footnote 7 continued)
a dn. Joh. Bernoullio geometris publice propositi, una cum solutione sua problematis alterius ab eodem postea propositi. A French translation is available in [167], pp. 351-358.

[8] He also noted that "l'Hôpital, Huygens were he alive, Hudde if he had not given up such pursuits, Newton if he would take the trouble" could also have solved the problem. In fact, Newton had published his answer anonymously in the January 1697 issue of the *Philosophical Transactions*; the paper was republished anonymously in the same issue of the *Acta*, see Section 2.3 and Section 2.4.

[9] Johann Bernoulli says in the *Letter to Basnage* [22] that he did not know about Galilei's considerations when posing his problem and that he had only learned of Galilei later from Leibniz, a claim that sounds doubtful on account of the celebrity of Galilei and of his *Dimostrazioni*.

1.1 PROPOSITION. *We have*

(1) among the piecewise-linear paths joining two given points in space the one with minimal length is the segment joining the two points;
(2) in an isosceles triangle the median, the perpendicular to the base and the bisectrix of the vertex angle agree;
(3) given a straight line r and a point P outside r there exists a unique point in r of minimal distance from P: It is the intersection of P with the perpendicular to r through P.

1.2.1 Fermat's Principle of Least Time

In the *Optics* of Euclid (325-265 BC), we find the by now familiar *reflection law of light*: If a light ray is sent toward a mirror, then the angle of incidence equals the angle of reflection[10], $\theta_i = \theta_r$ in Figure 1.2[11]. In fact this holds not only for a flat mirror but for a curved mirror as well (angles, of course, being measured with respect to the tangent line at the point of reflection).

Heron's principle

Heron of Alexandria[12] observed then, in his book on mirrors *Catoptrica*, that the reflection principle is a mathematical consequence of a minimum principle (probably the first occurrence of a *minimum principle* in mathematical physics) now called

HERON'S PRINCIPLE. *In a homogeneous medium, light travels from a source to a receiver by taking the shortest path.*

A simple consequence of Heron's principle (and of Proposition 1.1) is that *in the absence of obstacles light travels straight* and

REFLECTION LAW FOR PLANE MIRRORS. *A ray of light is reflected by a plane mirror in such a way that it remains in the orthogonal plane to the mirror determined by the ray itself and with an angle of reflection equal to the angle of incidence.*

In fact minimality implies that the ray has to lie in the orthogonal plane to the mirror through A and B, and, see Figure 1.2, if at R we have $\theta_i = \theta_r$ then the length

---

[10] The law was known also to Archimedes (287-212 BC) who had proved it by symmetry: If $\theta_i \neq \theta_r$ for instance $\theta_i > \theta_r$ then, by inverting the direction of the ray, we would get $\theta_r > \theta_i$.

[11] More precisely, the incident and reflected ray lie in the same plane through the source and the target and orthogonal to the mirror and $\theta_i = \theta_r$, see next paragraph.

[12] Heron of Alexandria was an encyclopedic scholar who wrote mainly about geometry and mechanics mixing approximate and rigorous procedures. Not much is known about him — determining the period in which he lived has been one of the most debated question in the mathematical historiography —; with sufficient certitude we know that he lived between 100 BC and 100 AC.

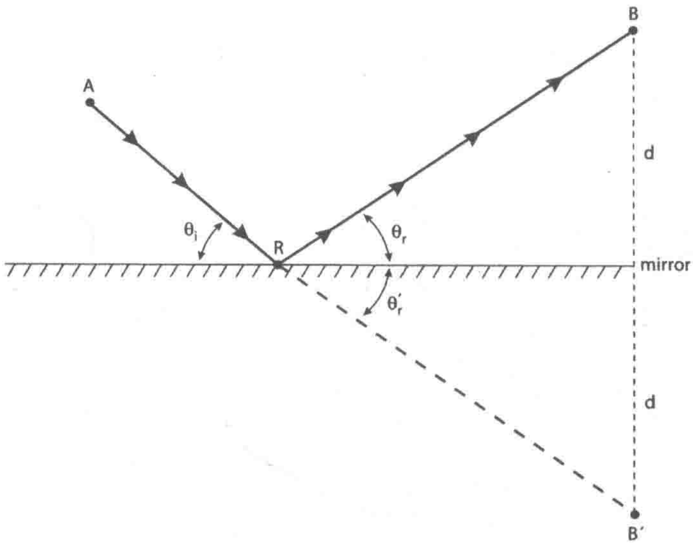

Fig. 1.2 The reflection principle and Heron's minimum length path.

of the path ARB is smaller than any other path AP followed by PB for any P on the mirror.

Fermat's principle

The *refraction* of light when passing from air to water can also be explained or formulated in terms of a minimum principle, but not as consequence of Heron's principle. Quite some time was needed to get to the point. Attempts to formulate a mathematical description of refraction can be traced as far back as Ptolemy (85-165) and, later, Kepler (1571-1630) (*Dioptrice*, 1611), but it was not until the Seventeenth century that the question was really tackled.

On the basis of experimental evidence in 1621 Willebrord Snell (Snellius) (1580-1626) formulated the following law, see Figure 1.3,

SNELL'S LAW. *The sines of the angles θ_i and θ_r that the incident and refracted rays make with the normal to the interface between two different media are proportional,*

$$\frac{\sin\theta_i}{\sin\theta_r} = \text{constant},$$

where the constant, now called the relative index of refraction, is characteristic of both media on either side of the interface.

Snell observed that if the medium 2 is denser than medium 1 (as for a light ray travelling from air to water) then the *constant* is greater than one. That is, $\sin\theta_i >$