TOPOLOGICAL METHODS IN GALOIS REPRESENTATION THEORY

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To Carolyn, Anna, Nina, and Daniel

Preface

This volume studies, from several viewpoints, the representation theory of finite groups which happen to be the Galois groups of finite extensions of fields. In particular, it is concerned with the construction of invariants of such Galois representations.

At the mention of invariants (or characteristic classes, in the topological terminology), an algebraic topologist would at once think of the more-than-adequate theory of Stiefel-Whitney and Chern classes and so might consider the matter closed. True, the methods of algebraic topology are designed for, and have been largely successful in, the process of constructing invariants. However, topology generally presents the seeker with invariants of Galois representations which are natural for all homomorphisms of such groups, which is much more than one insists upon when studying these representations qua Galois representations. It was this novelty which aroused my interest in this subject and, within this volume, I hope to give some bona fide examples in which a modicum of algebraic topology is extremely useful—perhaps even essential.

The first four chapters of this book are concerned with characteristic classes (of Galois representations) whose values lie in mod 2 Galois cohomology. The topic treated is the relationship, first discovered by Jean-Pierre Serre, between algebraic and topological characteristic classes of a Galois representation. That is, as explained in Chapter 2, an orthogonal Galois representation may be considered as giving rise to a bilinear form. The Hasse-Witt classes of this bilinear form turn out to be related to the Stiefel-Whitney classes of the representation. In Chapters 3 and 4 we derive Serre's formula and generalizations of it due to A. Frohlich and B. Kahn. These results we prove by methods that differ from the original ones and which require a modest amount of topology. For example, we develop the Koslowski transfer ab initio, in Chapter 4, in the category of topological spaces. In this setting the process is simpler and more general than the algebraic manner in which Bruno Kahn rediscovered it.

In preparation for later applications, Chapter 1 gives a brief introduction to the abelian cohomology of groups, and Chapter 2 does the same for the nonabelian theory. In those chapters several examples of cohomology rings are calculated. In particular, it is at this point that we collect all the specific cohomological data about dihedral and quaternion groups which will be useful later.

Chapters 6 and 7 are concerned with the construction of invariants of Galois representations in local class field theory. These chapters culminate in a new, essentially local, construction of the local root numbers, which give a local/global factorization of the Artin root number. In addition, as a necessary preliminary step, we construct the orthogonal local root numbers in Chapter 6, section 2, by a new, ad hoc method, involving the Witt group of nonsingular, symmetric, bilinear forms. This feature provides a very satisfactory point of contact between the material of Chapter 3 and that of Chapter 7.

Chapter 6 introduces the canonical form, which I have christened Explicit Brauer Induction, of Brauer's induction theorem. This involves more serious topology, in the form of the Lefschetz Fixed-Point Theorem. This chapter also derives a natural presentation for the representation ring of a finite group in a form which is suitable for the procedure of promoting invariants of abelian Galois representations to give invariants of arbitrary Galois representations. The problem of finding such a presentation is a very natural one and was posed by Jean-Pierre Serre. The construction of the local root number is an excellent example of this procedure in action, and I imagine that the formal nature of the argument will eventually render the technique useful in other contexts.

Finally, I will describe the role of Chapter 5. This chapter treats hard core stable homotopy theory that is not essential to the understanding of the later chapters. If the details are too unfamiliar, I recommend merely the reading of the statements of the main results and the scrutiny of the numerous attendant examples.

In Chapter 5 a result is proved which concerns the stable homotopy classes of maps between classifying spaces of groups. This result leads inexorably to the discovery of the Explicit Brauer Induction formulae, at least in the $I(G)$-adically completed representation ring. All this is described in Chapter 5, with many illustrative examples, and in Chapter 6, section 1. Therefore, Chapter 5 is an example of a result in stable homotopy theory which leads the way to a serious, new result in representation theory and thence to a serious application in number theory. I have included Chapter 5 to emphasize the novelty of this transpiration.

This volume began as lecture notes for a graduate course I gave at the University of Western Ontario during 1985 to 1986. The lecture notes contained Chapters 1–5 in essentially their current form and a far less satisfactory version of Chapters 6–7, in which the representation rings had to be completed and all invariants considered had to be continuous.

Throughout the book I have tried to give sufficient background on the topological prerequisites so that the energetic reader could pursue the details further. On this basis I believe that the reader who has experienced a graduate course on introductory algebraic topology will find this book accessible. On the algebraic and number-theoretic side, I have tried to be more complete, partly because of the constitution of my original audience.

I am very grateful to the University of Western Ontario for granting me a sabbatical year to finish this book. I was fortunate to enjoy the hospitality of

the Centre de Recherches Mathématiques, Université de Montréal, and of the Mathematical Sciences Research Institute, Berkeley, during the final stages. I have attempted to embellish the start of each chapter with a quotation of idiosyncratic aptness, in that regard I am very grateful to Nancy Z. Tausky for providing me with the translation of the original lines of Belshazzar to Daniel (from "Cleanness" 1633–1640), and likewise, I would like to thank Mikael Runsten and Udo Zander for the original words of the poem by F. M. Franzén. Finally, I am deeply indebted to Catharine Leggett for typing the manuscript.

Victor Snaith

Hamilton, Ontario
April 1988

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Chapter One

Abelian Cohomology of Groups

It's like a book, this bloomin' world.
Which you can read and care for just so long,
But presently you feel that you will die
Unless you get the page you're readin' done,
An' turn another—likely not so good;
But what you're after is to turn 'em all.
—*RUDYARD KIPLING,*
"*Sestina of the Tramp-Royal*" (1896)

In this chapter we first review the basic definitions of group cohomology with abelian coefficients, both continuous and discrete. Then we consider explicit formulae in low dimensions for applications such as products and the transfer (or corestriction) map. We introduce the usual basic concepts, for example, the long exact sequence and the homology/cohomology relationship. Our primary goal is to come away from this chapter with a few specific cohomology rings at our disposal—as well as transfer techniques such as the double coset formula. Transfer techniques are not only technically useful to us at this point; we will need to depend on them when, in Chapter 5, we encounter the double coset formula in stable homotopy theory.

After we have computed the cohomology rings of the cyclic groups, the dihedral group of order eight (with mod 2 coefficients) and (additively) the cohomology of the generalized quaternion groups, certain acts of faith will be required. These take the form of belief in the Stiefel-Whitney classes, Chern classes, and in the properties of spectral sequences. I have taken the view that faith knows no limits! Accordingly, with the briefest review of such things we are able to conclude this chapter with the computation of the mod 2 cohomology rings of dihedral and generalized quaternion groups and the integral cohomology ring of the dihedral group of order eight.