Introduction to

physical mathematics

P.G. HARPER

D.L. WEAIRE

Introduction to

physical mathematics

P.G. HARPER

Professor of Theoretical Physics, Heriot-Watt University, Edinburgh

D.L. WEAIRE

Erasmus Smith's Professor, Department of Pure and Applied Physics
Trinity College, Dublin

The right of the University of Cambridge to print and sell all manner of books was granted by Henry VIII in 1534. The University has printed and published continuously since 1584.

CAMBRIDGE UNIVERSITY PRESS

Cambridge

London New York New Rochelle

Melbourne Sydney

Published by the Press Syndicate of the University of Cambridge
The Pitt Building, Trumpington Street, Cambridge CB2 1RP
32 East 57th Street, New York, NY 10022, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1985

Printed in Great Britain at the University Press, Cambridge

Library of Congress catalogue card number: 84–11411

British Library cataloguing in publication data

Harper, P.G.
Introduction to physical mathematics.
1. Mathematical physics
I. Title II. Weaire, D.L.
530.1'5 QC20
ISBN 0 521 26278 hard covers
ISBN 0 521 26908 3 paperback

TP

Preface

The theoretical side of physical science holds up a mathematical mirror to nature. It seeks to find in the infinite variety of physical phenomena the few basic laws and relationships which underlie them. A secondary goal is the expression of these relations in efficient and transparent language.

After Newton had shown the power of this method, the eighteenth and nineteenth centuries saw its steady advance, hand-in-hand with experiment. At the end of the nineteenth century there was a crisis in physics – a widening gulf between theory and experiment – but, when Einstein emerged to resolve it, the new physics was still based on the old mathematics. It was simply used in surprising new ways. So it remains today, to a large extent, whatever educational theorists may tell us. Newton would not be greatly puzzled by the mathematics of Schrödinger's Equation.

On the other hand, the rapid development of computers is certainly changing our attitude to mathematics. This is obvious in the case of straightforward numerical calculations, but it extends also to the simulation of complex systems, the manipulation of algebra and even the proving of theorems. Applied mathematics is the art of the possible, and computers have widened its scope enormously. They are not just 'number-crunchers'. Nor are they available only to specialists. Most students today enjoy access to a powerful computer system, and many are skilled programmers at an early age.

Today's physical scientist needs both a feeling for the power of traditional analytic methods in relation to the physical world and an appreciation of modern computational methods. University curricula which rigidly separate mathematics, physics and computer science do not serve him well. At some stage, preferably at the beginning, these subjects should be brought together. This is what we have tried to do.

Our subject matter divides naturally into three parts. Elementary aspects of vectors, matrices and functions are introduced in chapters 1–11. The use of calculus and various approximate methods in solving physical problems, particularly those which involve differential equations, is covered in chapters 12–24. Chapters 25–34 introduce physical fields and the associated partial differential equations. Our objective throughout is the development of qualitative understanding and practical know-how, rather than rigour and completeness.

At various points in the text we have gone right back to basics, to explain things which university and college students have surely met before, such as differentiation. Most lecturers find that they must do this in an introductory course, whatever the curriculum may say, in order to bring all of the class up to a similar level of preparedness. Moreover, even the simplest operations raise many questions when we consider the pitfalls of their practical applications.

At Heriot-Watt, lectures based on this book run in parallel to a conventional course in pure mathematics (mostly calculus) in the first year, forming a bridge between the mathematics and physics courses. The class is given plenty of time to explore the exercises and encouraged to do so critically and creatively.

These exercises are indeed an integral part of the course and should be studied regularly. Some of the numerical ones are quite open-ended, since already students are using a great variety of computing hardware and it is hard to say what they may be using within a few years. Ideally, they should get some practice in the use of back-of-envelope arithmetic, hand calculators, and computers, including perhaps some library subroutines for numerical analysis. Some of the exercises make good material for a classroom discussion, reinforcing the message that applied mathematics is not a collection of cut-and-dried procedures but a flexible framework within which physical systems can be described in a variety of ways. This is not just an introduction – it is an *invitation*.

P.G. Harper
D.L. Weaire

Some notes on notation

The mathematical notation that we use is quite traditional. The following notes may be helpful in resolving some ambiguities.

Symbol **Meaning and Notes**

$\rightarrow$ *tends to*

The operation of taking a limit is the foundation of the calculus. The limit symbol should not be confused with $\sim$ (see below). Occasionally we also use an arrow to indicate replacement (so that $a \rightarrow b$ means 'a is replaced by b'), or displacement from one point to another.

$\sim$ *behaves as*

This always applies in some limit (sometimes implied rather than stated). For example, '$f(x) \sim g(x)$ as $x \rightarrow \infty$' means

$$\frac{f(x)}{g(x)} \rightarrow 1 \text{ as } x \rightarrow \infty.$$

$\approx$ *approximately equals*

Strictly speaking, this is rather meaningless unless there is some statement of the magnitude of the departure from equality (the error). Nevertheless it is widely used more casually to indicate the replacement of an exact value or formula by one which is not exact but of acceptable accuracy for the purpose at hand.

O( ) *of order…*

Again a limit is involved. For example, $f(x) = O(x^3)$ as $x \to 0$ means $x^{-3}f(x) \to$ *a finite limit*, as $x \to 0$. It thus has a similar meaning to that of $\sim$, but less strict since the finite limit does not have to be unity. In particular, it may be zero.

Δ (*Greek capital delta*) *increment of*

This is used here and in many introductory physics texts for an increment of a variable. Usually a *small* increment is implied, and often the limit is eventually taken in which all increments go to zero. This leads in some cases to differential relations, via

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right)$$

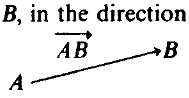
and in others to integral relations.

v (*bold typeface*) *a vector quantity*

A physical vector, or a column vector in matrix theory. In the latter, it is usual to reserve lower case letters for vectors, and capitals for matrices, except where tradition dictates otherwise!

$\overrightarrow{AB}$ *vector which represents the line AB*

In the use of vectors in association with geometrical constructions, this alternative notation denotes the vector of magnitude equal to the length of AB, in the direction indicated.



Σ (*Greek capital sigma*) *'sum'*

This has the usual meaning of a summation, but physical scientists often use it without the indication of the summation labels and range which are strictly required. Thus $\sum_{i=1}^{N} m_i x_i$ might be written $\sum mx$ whenever its meaning is obvious from the context.

The Greek alphabet

In the interests of clarity and to avoid ambiguity, Greek symbols are often invoked in physical mathematics, to supplement the Roman alphabet. Many have traditional connotations – λ for wavelength, ν for frequency, ρ for density... . Mercifully, the use of other alphabets or typefaces, such as gothic, seems to be dying out.

	lower case	*capital*	
alpha	α	A	
beta	β	B	
gamma	γ	Γ	
delta	δ	Δ	(see notes above)
epsilon	ε	E	
zeta	ζ	Z	
eta	η	H	
theta	θ	Θ	
iota	ι	I	
kappa	κ	K	
lambda	λ	Λ	
mu	μ	M	
nu	ν	N	
xi	ξ	Ξ	
omicron	o	O	
pi	π	Π	
rho	ρ	P	
sigma	σ	Σ	(see notes above)
tau	τ	T	
upsilon	υ	Y	
phi	ϕ	Φ	
chi	χ	X	
psi	ψ	Ψ	
omega	ω	Ω	

The symbol ∇ is usually called **'grad'** but, somewhat confusingly, ∇^2 is called 'del-squared'! These stand for operators (chapters 28, *et seq.*).

Contents

Some elementary computer programs, together with answers to the exercises, are contained in a supplementary booklet. These may be obtained from the Department of Theoretical Physics, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS.

1 Introduction

Most physical laws express numerical relations between quantities which can be independently measured, such as the mass of a body, its acceleration and the force which is applied to it. Ultimately, they are established or refuted by experiment. The range of their validity is determined by the range of practicable experiments. Their generality is always in question, and physicists continually seek new insights in the breakdown of old theories.

It is important therefore to distinguish between physical *laws*, which are provisional and approximate (since, in principle, we expect to find circumstances in which they do not apply), and other mathematical relationships which are merely conventional *definitions*, such as 'momentum equals mass times velocity'. These cannot be overturned, although there may be a time or a place in which they are not useful.

Given a problem to solve, we make the transition to mathematics by choosing appropriate physical laws and definitions. For the purposes of mathematical manipulation we may provisionally regard this formulation as exact, but in practice we will soon encounter uncertainties of two kinds.

First, if we wish to use experimentally measured quantities as numerical input to our calculations, as must ultimately be the case, we should recognise that every measurement involves some degree of uncertainty. The word 'error' is commonly used for this, which is unfortunate, because it need not be the result of any mistake or misjudgement, but may simply follow from the limited accuracy of the available measuring apparatus. We should be aware of the magnitude of this error and try to trace its effects through our calculation, to see what bearing it has on the final result or output. This can then be assigned some estimate of uncertainty or 'error bar'.

Secondly, any numerical calculation that we perform entails further errors. These may arise from the round-off error of the computer or calculator (since only a finite number of digits can be retained), or the approximations of the numerical methods which are used.

In chapter 2 we shall look at errors in more detail. In the rest of the chapters we shall talk about them only when some interesting point arises, e.g. when there is a possibility of large errors in a particular numerical method. However, in every real application of mathematics this aspect must not be overlooked.

It must also be borne in mind that physical quantities are expressed in terms of units, and only rarely do we meet dimensionless quantities which do not require them. Even in a mathematically oriented course units must be respected.

On the other hand, whenever one is concentrating on the mathematical aspects of a theory the units may often be disregarded, as we do from time to time. The closer one gets to real applications the more important it is to remember that all input values and all final results should have clearly stated units. If one works within a consistent scheme of units such as the SI system one can often ignore units at intermediate stages in a calculation, and assign the obvious units to the final result.

So the full result of a measurement or calculation should look something like the following (for an acceleration):

$$a = 6.32 \pm 0.02\,\mathrm{m\,s^{-2}}. \tag{1.1}$$

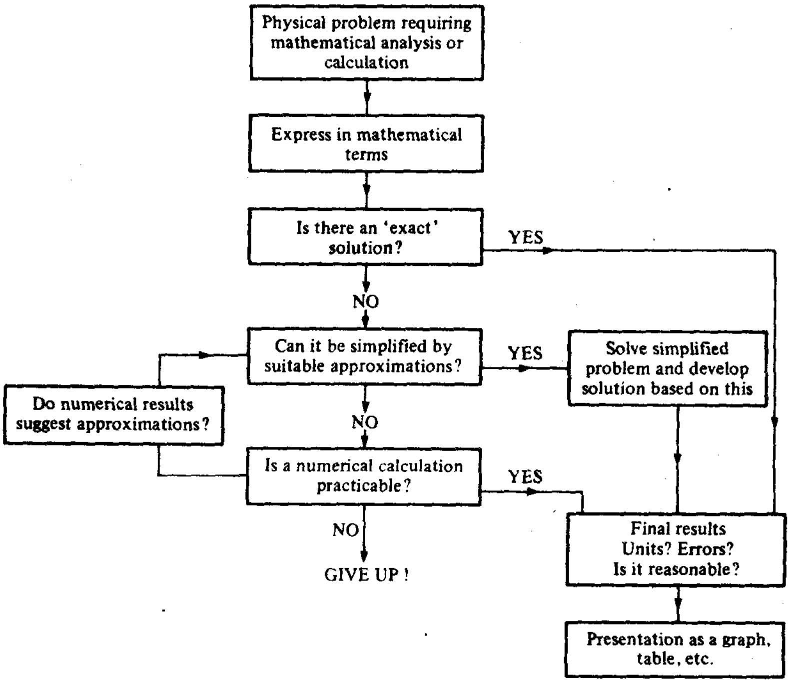
Without some reference to units, this would be meaningless. Without the error estimate ± 0.02, it would still be acceptable, but would carry the implication that the result 6.32 was expected to be correct to within 0.005, i.e. that errors are unlikely to affect the last digit which is retained. In either case, some thought must be given to the number of significant figures which are retained. The above might have been the result of a computer calculation which printed out 6.322 371. The error estimate causes us to round off this number by the elimination of the last four digits, because they are insignificant. It would have been pointless to include more digits, in view of the error associated with the result. Indeed if the error had not been stated, extra digits would have been quite misleading. People often forget to 'prune' their results in this way. An amusing example is provided by the Laws of Rugby Football in which, for the metrically minded referee, it is recommended that the ball should have 'a pressure equivalent to 0.6697–.7031 kilograms per square centimetre at sea level'. Some worthy official worked this out by converting the old fashioned '9–10 pounds per square inch', but would have done well to drop at least two figures from his results.

What about the actual mathematical methods – how do they proceed? You should try to develop a flexible, balanced attitude to this, and remember that there are at least two goals – quantitative description and qualitative understanding. Prescribed questions for homework and examinations may call for some obvious technique which you have just learned. But in reality, any given problem in physical mathematics can be attacked from many directions. Are there powerful theorems which will transform or reduce it to something simpler? Is there an 'exact' analytical solution? (We shall discuss the meaning of this in chapter 10.) Is there an exact solution to a related problem, which can be used as a starting approximation? Having exhausted this line of enquiry, the applied mathematician nowadays takes his problem to the computer, in the form most amenable to calculation. Again, there may be a choice of methods. Increasingly, this is done at quite an early stage because today's machines can do wonders with even clumsy and elementary numerical methods. Often, the results of this 'brute force' approach will inspire analytical methods or approximations that could not be formulated in the first place. Of course, such a comprehensive view cannot be acquired all at once, but at every stage you should try to 'look over your shoulder' at methods you have already studied which relate to the subject at hand.

While exceptional individuals are completely comfortable with symbolic mathematics, most of us make visual images of even the most abstract relationships. Sketches are always helpful in this respect. We have included many of these, and the student is encouraged to do likewise. Computer graphics have begun to be very widely used as an aid to evaluating numerical output. Increasingly, moving images and stereographic projections are used. We have taken a very modest step in this exciting direction by including computer-generated graphical output wherever appropriate. Much of this was produced by the same few simple commands to a software package associated with an HP7470A Plotter, so similar plots can be easily generated and extended by students or lecturers. In the absence of a plotter, a simple 'line-printer plot' will often suffice.

Summary

1.1. Flow chart for practical mathematicians.



EXERCISES

1. Explain in about a paragraph the meaning of each of the following terms used in the text: *round-off, analytical, stereographic, software, qualitative, line-printer plot.*
2. Discuss the following assertion, taken from the preface: 'applied mathematics is the art of the possible'.
3. The population of a country is an integer with discontinuous changes, yet it is often described as if it were a continuous function of time. Discuss this, contrasting it with the case of the size of a family.

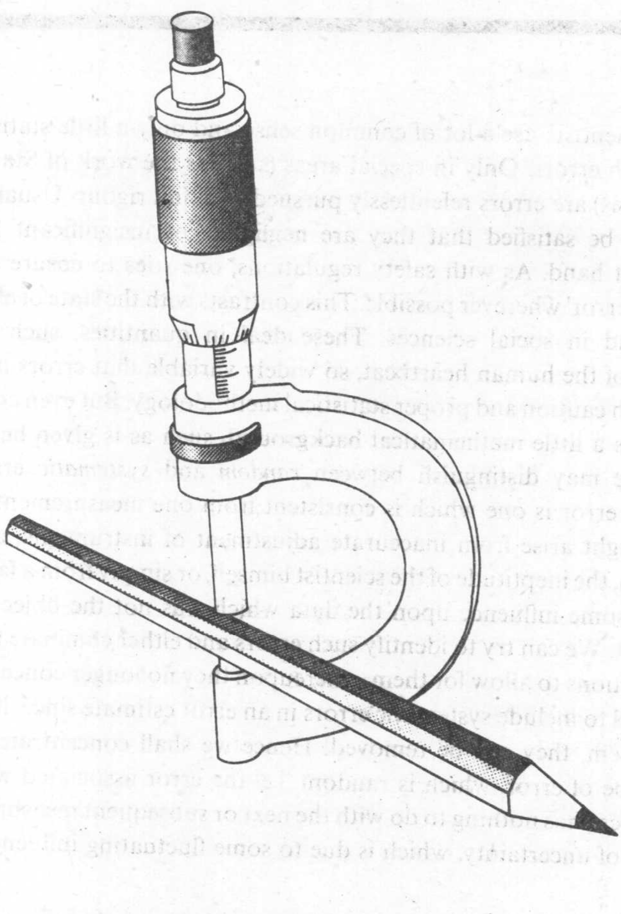
2
Errors

Physical scientists use a lot of common sense and only a little statistics in dealing with errors. Only in special areas (such as the work of Standards Laboratories) are errors relentlessly pursued with full rigour. Usually, it is enough to be satisfied that they are negligible or insignificant for the purposes at hand. As with safety regulations, one tries to ensure a large 'margin of error' wherever possible. This contrasts with the state of affairs in biology and in social sciences. These deal in quantities, such as the frequency of the human heartbeat, so widely variable that errors must be treated with caution and proper statistical methodology. But even common sense needs a little mathematical background, such as is given here.

First, we may distinguish between *random* and *systematic* errors. A systematic error is one which is consistent from one measurement to the next. It might arise from inaccurate adjustment of instruments, a faulty calibration, the ineptitude of the scientist himself, or simply from a failure to recognise some influence upon the data which was not the object of the experiment. We can try to identify such errors and either eliminate them or add corrections to allow for them, whereupon they no longer concern us. It is not usual to include systematic errors in an error estimate since, if we can identify them, they can be removed! Hence we shall concentrate on the second type of error, which is random, i.e. the error associated with one measurement has nothing to do with the next or subsequent measurements. This kind of uncertainty, which is due to some fluctuating influence upon the measurement, is recognised if we repeat one measurement several times and obtain slightly different results. Unlike the systematic error, random error should not affect the *average* of a sufficiently large number of measurements. This is the key to its reduction. How then are we to do this and how are we to express the final result?

To make this concrete, consider the measurement of the diameter of a standard pencil taken from stock. A micrometer can apparently measure this to within a few per cent of a millimetre. But in applying the gauge to the soft wood, we may compress it slightly. To the extent that we do this consistently, the error is systematic and could be identified by suitable experiments and so allowed for. There would, however, remain a variable part of the error, due to a variable force being applied to tighten the

2.1. Measurement of pencil diameter with a micrometer.



micrometer. This might be assumed to be random. If this uncertainty is greater than that associated with the instrument itself and its scale, it must take over as the main source of the error associated with the measurement.

The pencil width itself will still be variable. This could be explored simply by repeating the measurement on different parts of a single pencil, or on different pencils. In fundamental physics, life is simpler—it seems that every two electrons, for instance, are exactly the same, as far as we can tell. The only uncertainty in our knowledge of the mass or charge of an electron is that derived from the process of measurement. However, with pencils as with much else, we encounter the kind of intrinsic variability which becomes so serious in biological science. It will make our arguments more concrete if we concentrate upon this variability, supposing it to be much greater than the uncertainty of measurement.

Our procedure will then be to measure the diameters of a large number of individual pencils, and try to extract from these a standard value and uncertainty limits, rather as we did in eq. (1.1). What do we mean by a standard value? It is natural to take the average or mean value from our measurements as an estimate of the mean value $\langle x \rangle$ which would be obtained in the limit of an infinite number of such measurements. We can get an immediate feeling for the random deviations from the average value by plotting the distribution of measured diameters, as in fig. 2.2. Ideally, this is a

2.2(a). Distribution function for pencil diameters (schematic).