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P G Lemarié-Rieusset

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Preface

This book is a self-contained exposition of recent results on the Navier–Stokes equations, presented from the point of view of real harmonic analysis. A quarter of the book is an introduction to real harmonic analysis, where all the material we need in the book is introduced and proved (this part is based on a lecture given at Paris XI–Orsay in February-June 1998); the reader is assumed to have a basic knowledge of functional analysis, including the theory of the Fourier transform of tempered distributions. The other parts of the book are devoted to the Navier–Stokes equations on the whole space and include many recent results, such as the Koch and Tataru theorem on existence of mild solutions [KOCT 01], the results of Brandolese [BRA 01] and Miyakawa on the decay of solutions in space [MIY 00] or time (with Schonbek [MIYS 01]), many results on uniqueness (Chemin [CHE 99], Furioli, Lemarié–Rieusset and Terraneo [FURLT 00], Lions and Masmoudi [LIOM 98], May [MAY 02], Meyer [MEY 99] and Monniaux [MON 99]), results on Leray's self-similar solutions (Nečas, Ružička and Šverák [NECRS 96] and Tsai [TSA 98]), results on the decay of Lebesgue or Besov norms of solutions (Kato [KAT 90], Cannone and Planchon [CANP 00]), and the existence of solutions for a uniformly square integrable initial value [LEM 98b]. Older classical results are included, such as the existence of Leray weak solutions [LER 34], the uniqueness theorems of Serrin [SER 62] and Sohr and Von Wahl [WAH 85], Kato's theorems on the existence of mild solutions [FUJK 64], [KAT 84], [KAT 92], and the regularity criterion of Caffarelli, Kohn and Nirenberg [CAFKN 82].

Many proofs and statements are original. I tried to give general statements in the theorems and to remain in the setting of real harmonic analysis when proving the theorems. At some points, I have chosen not to give the shortest proofs, but to give proofs using only materials that are found in the limits of this book.

I am a newcomer in the vast realm of the theory of the Navier–Stokes equations, beginning to work seriously in this field in 1995, when I moved from the Université Paris XI-Orsay to the Université d'Évry and when G. Furioli and E. Terraneo began to prepare their theses with me.

At that time, I had no specific knowledge in the theory of PDEs, working rather in the field of real harmonic analysis. I was a specialist in wavelets, and before their invention by Meyer in 1985, I had worked on singular integrals, the Littlewood–Paley decomposition, and Besov spaces.

I took interest in the Navier–Stokes equations when Cannone finished his thesis [CAN 95], where the main tools were precisely wavelets, the Littlewood–Paley decomposition and Besov spaces. In February 1997, using Besov spaces, Furioli, Terraneo and I were able to prove uniqueness of solutions in the space $\mathcal{C}([0,T),(L^3(\mathbb{R}^3))^3)$ [FURLT 00]. Some monthes later, Meyer gave a simpler proof of this uniqueness result, using Lorentz spaces instead of Besov spaces [MEY 99].

I then taught at Université Paris XI-Orsay lecturing on the Navier–Stokes equations viewed from the point of view of real harmonic analysis, including introductory lessons on Besov and Lorentz spaces. Though I had heard about Lorentz spaces for fifteen years, this was still a brand new topic for me. The book is mainly based on my efforts to give a simple and efficient introduction to those technical spaces, in order to get efficient tools for proving inequalities. Thus, I have chosen to introduce Besov and Lorentz spaces through the discrete J-method of real interpolation, as the most simple and direct way to get sharp inequalities.

The efficiency of this approach may be seen in the chapter on Leray's self-similar solutions (Chapter 34), where we give a simplified proof of Tsai's results [TSA 98] and in the chapters on uniqueness of mild solutions (Chapters 27 and 28).

I owe the writing of this book to many people: H. Brezis, who asked me to write it; the members of the Department of Mathematics at Université d'Évry, who have created a very agreeable working environment; the Department of Mathematics at Université Paris-XI Orsay (especially, the Équipe d'Analyse Harmonique) who gave me the opportunity to lecture on the Navier–Stokes equations and thus to get a safer and more basic introductory point of view on this topic; my students and co-workers G. Furioli, R. May, E. Terraneo, E. Zahrouni and A. Zhioua who have helped me so much in the understanding of the Navier–Stokes equations; M. Cannone and F. Planchon who gave me their stimulating preprints on which so many chapters in this book are based; Y. Meyer who taught me so much and who took a constant interest in my work; and of course my wife and daughter who had to live in the same house with a monomaniac cyclothymic writer.

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Introduction

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Chapter 1

What is this book about?

There is a huge literature on the mathematical theory of the Navier-Stokes equations, including the classical books by R. Temam [TEM 77], O.A. Ladyzhenskaya [LAD 69] or P. Constantin and C. Foias [CONF 88]; a more recent reference is the book by P.L. Lions [LIO 96]. Modern references on mild solutions and self-similar solutions in the setting of $\mathbb{R}^3$ are the books by M. Cannone [CAN 95] and Y. Meyer [MEY 99]. Another useful reference is the book by W. von Wahl [WAH 85].

In this book, we shall examine the Navier–Stokes equations in d dimensions (especially in the case $d = 3$) in a very restricted setting: we consider a viscous, homogeneous, incompressible fluid that fills the entire space and is not submitted to external force. The equations describing the evolution of the motion $\vec{u}(t,x)$ of the fluid element at time t and position x are given by:

$$\begin{cases} \rho\, \partial_t \vec{u} = \mu\, \Delta \vec{u} - \rho\, (\vec{u}.\vec{\nabla})\, \vec{u} - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{1.1}$$

The *divergence free* condition $\vec{\nabla}.\vec{u} = 0$ expresses the incompressibility of the fluid. In equation (1.1), ρ is the (constant) *density* of the fluid, μ is the *viscosity* coefficient, and p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0. We may assume with no loss of generality that $\rho = \mu = 1$ (changing the unknown $\vec{u}(x,t)$ and $p(x,t)$ into $\vec{u}(\frac{\mu x}{\rho}, \frac{\mu t}{\rho})$ and $\frac{1}{\rho} p(\frac{\mu x}{\rho}, \frac{\mu t}{\rho})$).

Since $\vec{\nabla}.\vec{u} = 0$, equation (1.1) can be rewritten (as far as $\vec{u}$ is a regular function):

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{1.2}$$

which is a condensed form of

$$\begin{cases} \text{For } 1 \leq k \leq d,\ \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p \\ \sum_{l=1}^d \partial_l u_l = 0 \end{cases} \tag{1.2$'$}$$

Taking the divergence of (1.2), we obtain

$$\Delta p = -\vec{\nabla} \otimes \vec{\nabla}.(\vec{u}. \otimes \vec{u}) = -\sum_{k=1}^{d} \sum_{l=1}^{d} \partial_k \partial_l (u_k u_l) \tag{1.3}$$

Thus, we formally derive the equations

$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases} \tag{1.4}$$

where $\mathbb{P}$ is defined as

$$\mathbb{P}\vec{f} = \vec{f} - \vec{\nabla}\frac{1}{\Delta}(\vec{\nabla}.\vec{f}) \tag{1.5}$$

We study the Cauchy problem for equation (1.4) (looking for a solution on $(0,T) \times \mathbb{R}^d$ with initial value $\vec{u}_0$) and transform (1.4) into the integral equation

$$\begin{cases} \vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u})\, ds \\ \vec{\nabla}.\vec{u_0} = 0 \end{cases} \tag{1.6}$$

We consider weak solutions to equation (1.2), (1.4) or (1.6). In (1.2), we take the derivatives in the distribution sense; thus, (1.2) is meaningful as soon as $\vec{u}$ is locally square-integrable. We need extra information on $\vec{u}$ to give meaning to (1.4) or (1.6) (and, in some cases, to prove that the systems are equivalent to each other). Since we work on the whole space $\mathbb{R}^d$, the operators $e^{t\Delta}$ and $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}$ that we use to write (1.6) are convolution operators. Therefore, we put special emphasis on shift-invariant estimates; this means that we are going to work in functional spaces invariant under spatial translations.

Part 1 is devoted to the recalling of some (presumably) well-known results of harmonic analysis on some special spaces of functions or distributions, and on some convolution operators (fractional integration, Calderón–Zygmund operators, Riesz transforms, etc.). In Parts 2 to 6, we apply those tools to the study of the Cauchy problem for the Navier–Stokes equations: Part 2 presents some general shift-invariant estimates for the Navier-Stokes equations; Part 3 reviews the classical existence results of Leray (weak solutions $\vec{u}$ such that $\vec{u} \in L^\infty((0,\infty),(L^2)^d)$, $\vec{\nabla} \otimes \vec{u} \in L^2((0,\infty),(L^2)^{d^2})$ [LER 34]) and Kato and Fujita (mild solutions in $\mathcal{C}([0,T],(H^s)^d), s \geq d/2-1$ [FUJK 64], or in $\mathcal{C}([0,T],(L^p)^d), p \geq d$ [KAT 84]); Part 4 and 5 describe some recent results on mild solutions (generalizations of Kato's results), including the theorem of Koch and Tataru on the existence of solutions for data in $BMO^{(-1)}$ [KOCT 01] and Cannone's theory of self-similar solutions [CAN 95]; Part 6 considers suitable solutions when $d=3$, the main tool is the local energy inequality of Scheffer [SCH 77] and the regularity criterion of Caffarelli, Kohn and Nirenberg

[CAFKN 82], with applications to the study of weak solutions with infinite energy.

1. Uniform weak solutions for the Navier–Stokes equations

We will focus on the invariance of equation (1.2) under spatial translations and dilations, as we consider the problem on the whole space $\mathbb{R}^d$. We begin by defining what we call a weak solution for the Navier–Stokes equations.

Definition 1.1: (Weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ *is a distribution vector field* $\vec{u}(t,x)$ *in* $(\mathcal{D}'((0,T) \times \mathbb{R}^d))^d$ *where*
a) $\vec{u}$ *is locally square integrable on* $(0,T) \times \mathbb{R}^d$
b) $\vec{\nabla}.\vec{u} = 0$
c) $\exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ $\partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p$

Notice that this is not the usual definition for weak solutions (as given in the books of Temam [TEM 77] or Von Wahl [WAH 85]).

Throughout the book, we use the following invariance of the set of solutions:
a) shift invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$, then $\vec{u}(t, x - x_0)$ is a weak solution on $(0,T) \times \mathbb{R}^d$;
b) dilation invariance: for $\lambda > 0$, $\frac{1}{\lambda}\vec{u}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$ is a solution on $(0, \lambda^2 T) \times \mathbb{R}^d$;
c) delay invariance: if $\vec{u}(t,x)$ is a weak solution of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^d$ and if $t_0 \in (0,T)$ then $\vec{u}(t + t_0, x)$ is a weak solution of the Navier-Stokes equations on $(0, T - t_0) \times \mathbb{R}^d$.

In order to use the space invariance, we introduce a more restrictive class of solutions:

Definition 1.2: (Uniformly locally square integrable weak solutions)

A weak solution of the Navier–Stokes equations on $(0,T) \times \mathbb{R}^d$ *is said to be uniformly locally square-integrable if for all* $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}^d)$ *we have* $\sup_{x_0 \in \mathbb{R}^d} \int\int |\varphi(x - x_0, t)\vec{u}(t,x)|^2 \, dx \, dt < \infty$.

Equivalently, $\vec{u}$ *is uniformly locally square-integrable if and only if for all* $t_0 < t_1 \in (0,T)$, *the function* $U_{t_0,t_1}(x) = (\int_{t_0}^{t_1} |\vec{u}(t,x)|^2 \, dt)^{1/2}$ *belongs to the Morrey space* L^2_{uloc}. *We then write* $\vec{u} \in \cap_{0<t_0<t_1<T} (L^2_{uloc,x} L^2_t((t_0,t_1) \times \mathbb{R}^d))^d$.

We now explain the utility of introducing uniform weak solutions. In order to understand equation (1.2), we only need to assume that $\vec{u}$ is locally square-integrable on $(0,T) \times \mathbb{R}^d$. We need a stronger assumption to make sense of

(1.4), since $\mathbb{P}$ is a non local operator. More precisely, we need to make sense of $\vec{\nabla}(\frac{1}{\Delta}\vec{\nabla} \otimes \vec{\nabla}.\vec{u} \otimes \vec{u})$. In Part 2, Chapter 11, we prove the following:

Theorem 1.1: (Elimination of the pressure)

i) If $\vec{u}$ is uniformly locally square-integrable on $(0,T) \times \mathbb{R}^d$ (in the sense of Definition 1.2), then $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ is well defined in $(\mathcal{D}'((0,T) \times \mathbb{R}^d))^d$ and there exists a distribution $P \in \mathcal{D}'((0,T) \times \mathbb{R}^d)$ so that $\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) = \vec{\nabla}.(\vec{u}\otimes\vec{u}) + \vec{\nabla}P$. Thus, if $\vec{u}$ is a solution of (1.4), then it is a solution of (1.2).

ii) Conversely, if $\vec{u}$ is a uniformly locally square-integrable weak solution of (1.2), and if $\vec{u}$ vanishes at infinity in the sense that for all $t_0 < t_1 \in (0,T)$, we have

$$\lim_{R\to\infty} \sup_{x_0\in\mathbb{R}^d} \frac{1}{R^d} \int_{t_0}^{t_1} \int_{|x-x_0|<R} |\vec{u}|^2 \, dx \, dt = 0,$$

then $\vec{u}$ is a solution of (1.4).

The next step, when looking at such weak solutions, attempts to define the initial value problem and equation (1.6). It is easy to see that if we assume the solution $\vec{u}$ is uniformly locally square-integrable *up to the border* $t = 0$ (i.e., that for all $t_1 < T$, we have $(\int_0^{t_1} |u(t,x)|^2 \, dt)^{1/2} \in L^2_{uloc}(\mathbb{R}^d)$), then $\vec{u}_0$ is well defined. In Chapter 11, we prove more precisely the following theorem:

Theorem 1.2: (The equivalence theorem)

Let $\vec{u} \in \cap_{t_1<T}(L^2_{uloc,x}L^2_t((0,t_1) \times \mathbb{R}^d))^d$. Then, the following assertions are equivalent:

(A1) $\vec{u}$ is a solution of the differential Navier–Stokes equations

$$\begin{cases} \partial_t\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

(A2) $\vec{u}$ is a solution of the integral Navier–Stokes equations

$$\exists \vec{u}_0 \in (\mathcal{S}'(\mathbb{R}^d))^d \quad \begin{cases} \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u}) \, ds \\ \vec{\nabla}.\vec{u}_0 = 0 \end{cases}$$

2. Mild solutions

Given $\vec{u}_0 \in \mathcal{S}'(\mathbb{R}^d)$, in order to find a solution to (1.6), a natural approach is to iterate the transform $\vec{v} \mapsto e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{v}\otimes\vec{v}) \, ds$ and to find a fixed point $\vec{u}$ for this transform. This is the so-called Picard contraction method, already in use by Oseen at the beginning of the 20th century to establish the (local) existence of a classical solution to the Navier–Stokes equations for a regular initial value [OSE 27].

A simple approach to this problem is trying to find a subspace $\mathcal{E}_T$ of $L^2_{uloc,x}L^2_t((0,T)\times \mathbb{R}^d)$ so that the bilinear transform

$$B(\vec{u},\vec{v}) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{v})\, ds$$

is bounded from $\mathcal{E}_T^d \times \mathcal{E}_T^d$ to $\mathcal{E}_T^d$. Then, we may consider the space $E_T \subset \mathcal{S}'$ defined by $f \in E_T$ iff $f \in \mathcal{S}'$ and $(e^{t\Delta}f)_{0<t<T} \in \mathcal{E}_T$. We reach the following easy existence result:

Theorem 1.3: (The Picard contraction principle)
Let $\mathcal{E}_T \subset L^2_{uloc,x}L^2_t((0,T)\times\mathbb{R}^d)$ be such that the bilinear transform B is bounded on $\mathcal{E}_T^d$. Then:
a) If $\vec{u}\in\mathcal{E}_T^d$ is a weak solution of the Navier–Stokes equations: $\partial\vec{u} = \Delta\vec{u} - \mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})$ and $\vec{\nabla}.\vec{u}=0$, then the associated initial value $\vec{u}_0$ belongs to E_T^d.
b) Conversely, there exists a positive constant C such that for all $\vec{u}_0\in E_T^d$ satisfying $\vec{\nabla}.\vec{u}_0 = 0$ and $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} < C$ there exists a weak solution $\vec{u}\in\mathcal{E}_T^d$ of the Navier–Stokes equations associated to the initial value $\vec{u}_0$: $\vec{u} = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u}\otimes\vec{u})\, ds$.

Part 4 (Chapters 16 to 22) is devoted to examples of such spaces $\mathcal{E}_T$. The solutions we obtain through the Picard contraction principle are called mild solutions. (This is a slight abuse; the notion of mild solutions was introduced by Browder [BRO 64] and Kato [KAT 65] in an abstract setting; we use a simplified definition because we work in a special simple setting).

We call a space $\mathcal{E}_T$ if we may apply the Picard contraction principle as an *admissible path space* for the Navier–Stokes equations, and the associated space E_T as an *adapted value space*. Thus, the "rule of the game" consists of identifying admissible path spaces and then characterizing the associated adapted value spaces. We find this rule more natural than the equivalent approach of Cannone [CAN 95] or Meyer [MEY 99] (which motivated ours): those authors start from the adapted value spaces before, specifying, when necessary, associated admissible path spaces.

Classical admissible spaces are provided by the L^p theory of Kato [KAT 84] and Weissler [WEI 81] (see Chapter 15):
– for $d<p<\infty$, $\mathcal{C}([0,T],L^p)$ is admissible with associated adapted space $L^p(\mathbb{R}^d)$;
– for $p=d$, the space

$$\{f\in\mathcal{C}([0,T],L^d)\ /\ \sup_{0<t<T}\sqrt{t}\|f\|_{L^\infty(dx)}<\infty \text{ and } \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(dx)}=0\}$$

is admissible with associated adapted space $L^d(\mathbb{R}^d)$;
– for $T=\infty$ (i.e., for global solutions in L^d), we use the admissible space

$$\{f\in\mathcal{C}(\mathbb{R}^+,L^d)/\sup_{0<t}\|f\|_{L^d(dx)}<\infty,\ \sup_{0<t}\sqrt{t}\|f\|_{L^\infty(dx)}<\infty, \lim_{t\to 0}\sqrt{t}\|f\|_{L^\infty(dx)}=0\}$$

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