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Serge Lang

Differential and Riemannian Manifolds

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Serge Lang
Department of Mathematics
Yale University
New Haven, CT 06520
USA

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Preface

This is the third version of a book on differential manifolds. The first version appeared in 1962, and was written at the very beginning of a period of great expansion of the subject. At the time, I found no satisfactory book for the foundations of the subject, for multiple reasons. I expanded the book in 1971, and I expand it still further today. Specifically, I have added three chapters on Riemannian and pseudo Riemannian geometry, that is, covariant derivatives, curvature, and some applications up to the Hopf–Rinow and Hadamard–Cartan theorems, as well as some calculus of variations and applications to volume forms. I have rewritten the sections on sprays, and I have given more examples of the use of Stokes' theorem. I have also given many more references to the literature, all of this to broaden the perspective of the book, which I hope can be used among things for a general course leading into many directions. The present book still meets the old needs, but fulfills new ones.

At the most basic level, the book gives an introduction to the basic concepts which are used in differential topology, differential geometry, and differential equations. In differential topology, one studies for instance homotopy classes of maps and the possibility of finding suitable differentiable maps in them (immersions, embeddings, isomorphisms, etc.). One may also use differentiable structures on topological manifolds to determine the topological structure of the manifold (for example, à la Smale [Sm 67]). In differential geometry, one puts an additional structure on the differentiable manifold (a vector field, a spray, a 2-form, a Riemannian metric, ad lib.) and studies properties connected especially with these objects. Formally, one may say that one studies properties invariant under the group of differentiable automorphisms which preserve the additional structure. In differential equations, one studies vector fields and their integral curves, singular points, stable and unstable manifolds, etc. A certain number of concepts are essential for all three, and are so basic and elementary that it is worthwhile to collect them together so that more advanced expositions can be given without having to start from the very beginnings.

It is possible to lay down *at no extra cost* the foundations (and much more beyond) for manifolds modeled on Banach or Hilbert spaces rather than finite dimensional spaces. In fact, it turns out that the exposition gains considerably from the systematic elimination of the indiscriminate use of local coordinates $x_1, \ldots, x_n$ and $dx_1, \ldots, dx_n$. These are replaced by what they stand for, namely isomorphisms of open subsets of the manifold on open subsets of Banach spaces (local charts), and a local analysis of the situation which is more powerful and equally easy to use formally. In most cases, the finite dimensional proof extends at once to an invariant infinite dimensional proof. Furthermore, in studying differential forms, one needs to know only the definition of multilinear continuous maps. An abuse of multilinear algebra in standard treatises arises from an unnecessary double dualization and an abusive use of the tensor product.

I don't propose, of course, to do away with local coordinates. They are useful for computations, and are also especially useful when integrating differential forms, because the $dx_1 \wedge \cdots \wedge dx_n$ corresponds to the $dx_1 \cdots dx_n$ of Lebesgue measure, in oriented charts. Thus we often give the local coordinate formulation for such applications. Much of the literature is still covered by local coordinates, and I therefore hope that the neophyte will thus be helped in getting acquainted with the literature. I also hope to convince the expert that nothing is lost, and much is gained, by expressing one's geometric thoughts without hiding them under an irrelevant formalism.

It is profitable to deal with infinite dimensional manifolds, modeled on a Banach space in general, a self-dual Banach space for pseudo Riemannian geometry, and a Hilbert space for Riemannian geometry. In the standard pseudo Riemannian and Riemannian theory, readers will note that the differential theory works in these infinite dimensional cases, with the Hopf–Rinow theorem as the single exception, but not the Cartan–Hadamard theorem and its corollaries. Only when one comes to dealing with volumes and integration does finite dimensionality play a major role. Even if via the physicists with their Feynman integration one eventually develops a coherent analogous theory in the infinite dimensional case, there will still be something special about the finite dimensional case.

One major function of finding proofs valid in the infinite dimensional case is to provide proofs which are especially natural and simple in the finite dimensional case. Even for those who want to deal only with finite dimensional manifolds, I urge them to consider the proofs given in this book. In many cases, proofs based on coordinate free local representations in charts are clearer than proofs which are replete with the claws of a rather unpleasant prying insect such as Γ^i_{jkl}. Indeed, the bilinear map associated with a spray (which is the quadratic map corresponding to a symmetric connection) satisfies quite a nice local formalism in charts. I think the local representation of the curvature tensor as in Proposition 1.2 of Chapter IX shows the efficiency of this formalism and its superiority over local coordinates. Readers may also find it instructive to compare the proof of Proposition 2.6 of Chapter IX concerning the rate of growth of Jacobi fields with more classical ones involving coordinates as in [He 78], pp. 71–73.

Of course, there are also direct applications of the infinite dimensional case. Some of them are to the calculus of variations and to physics, for instance as in Abraham–Marsden [AbM 78]. It may also happen that one does not need formally the infinite dimensional setting, but that it is useful to keep in mind to motivate the methods and approach taken in various directions. For instance, by the device of using curves, one can reduce what is a priori an infinite dimensional question to ordinary calculus in finite dimensional space, as in the standard variation formulas given in Chapter IX, §4.

Similarly, the proper domain for the geodesic part of Morse theory is the loop space (or the space of certain paths), viewed as an infinite dimensional manifold, but a substantial part of the theory can be developed without formally introducing this manifold. The reduction to the finite dimensional case is of course a very interesting aspect of the situation, from which one can deduce deep results concerning the finite dimensional manifold itself, but it stops short of a complete analysis of the loop space. (Cf. Boot [Bo 60], Milnor [Mi 63].) This was already mentioned in the first version of the book, and since then, the papers of Palais [Pa 63] and Smale [Sm 64] appeared, carrying out the program. They determined the appropriate condition in the infinite dimensional case under which this theory works.

In addition, given two finite dimensional manifolds X, Y it is fruitful to give the set of differentiable maps from X to Y an infinite dimensional manifold structure, as was started by Eells [Ee 58], [Ee 59], [Ee 61], and [Ee 66]. By so doing, one transcends the purely formal translation of finite dimensional results getting essentially new ones, which would in turn affect the finite dimensional case.

Foundations for the geometry of manifolds of mappings are given in Abraham's notes of Smale's lectures [Ab 60] and Palais's monograph [Pa 68].

For more recent applications to critical point theory and submanifold geometry, see [PaT 88].

One especially interesting case of Banach manifolds occurs in the theory of Teichmuller spaces, which, as shown by Bers, can be embedded as submanifolds of a complex Banach space. Cf. [Ga 87], [Vi 73].

In the direction of differential equations, the extension of the stable and unstable manifold theorem to the Banach case, already mentioned as a possibility in the earlier version of this book, was proved quite elegantly by Irwin [Ir 70], following the idea of Pugh and Robbin for dealing with local flows using the implicit mapping theorem in Banach spaces. I have included the Pugh–Robbin proof, but refer to Irwin's paper for the stable manifold theorem which belongs at the very beginning of the theory of ordinary differential equations. The Pugh–Robbin proof can also be adjusted to hold for vector fields of class H^p (Sobolev spaces), of importance in partial differential equations, as shown by Ebin and Marsden [EbM 70].

It is a standard remark that the C^∞-functions on an open subset of a euclidean space do not form a Banach space. They form a Fréchet space (denumerably many norms instead of one). On the other hand, the implicit function theorem and the local existence theorem for differential equations are not true in the more general case. In order to recover similar results, a much more sophisticated theory is needed, which is only beginning to be developed. (Cf. Nash's paper on Riemannian metrics [Na 56], and subsequent contributions of Schwartz [Sc 60] and Moser [Mo 61].) In particular, some additional structure must be added (smoothing operators). Cf. also my Bourbaki seminar talk on the subject [La 61]. This goes beyond the scope of this book, and presents an active topic for research.

I have emphasized differential aspects of differential manifolds rather than topological ones. I am especially interested in laying down basic material which may lead to various types of applications which have arisen since the sixties, vastly expanding the perspective on differential geometry and analysis. For instance, I expect the marvelous book [BGV 92] to be only the first of many to present the accumulated vision from the seventies and eighties, after the work of Atiyah, Bismut, Bott, Gilkey, McKean, Patodi, Singer, and many others.

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Added Comments, 1995. Immediately after the present book appeared in 1995, two other books also appeared which I wish to recommend very highly. One of them is the second edition of Gilkey's book *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index Theorem* (CRC Press, 1995). The other is the second edition of Klingenberg's *Riemannian Geometry* (Walter de Gruyter, 1995), which includes a nice chapter on the infinite dimensional Hilbert manifold of H^1-mappings, and several substantial applications to topology and closed geodesics on various compact manifolds.

Acknowledgments

I have greatly profited from several sources in writing this book. These sources include some from the 1960s, and some more recent ones.

First, I originally profited from Dieudonné's *Foundations of Modern Analysis*, which started to emphasize the Banach point of view.

Second, I originally profited from Bourbaki's *Fascicule de résultats* [Bou 69] for the foundations of differentiable manifolds. This provides a good guide as to what should be included. I have not followed it entirely, as I have omitted some topics and added others, but on the whole, I found it quite useful. I have put the emphasis on the differentiable point of view, as distinguished from the analytic. However, to offset this a little, I included two analytic applications of Stokes' formula, the Cauchy theorem in several variables, and the residue theorem.

Third, Milnor's notes [Mi 58], [Mi 59], [Mi 61] proved invaluable. They were of course directed toward differential topology, but of necessity had to cover ad hoc the foundations of differentiable manifolds (or, at least, part of them). In particular, I have used his treatment of the operations on vector bundles (Chapter III, §4) and his elegant exposition of the uniqueness of tubular neighborhoods (Chapter IV, §6, and Chapter VII, §4).

Fourth, I am very much indebted to Palais for collaborating on Chapter IV, and giving me his exposition of sprays (Chapter IV, §3). As he showed me, these can be used to construct tubular neighborhoods. Palais also showed me how one can recover sprays and geodesics on a Riemannian manifold by making direct use of the fundamental 2-form and the metric (Chapter VII, §7). This is a considerable improvement on past expositions.

Finally, in the direction of differential geometry, I found Berger–Gauduchon–Mazet [BGM 71] extremely valuable, especially in the way they lead to the study of the Laplacian and the heat equation. This book has been very influential, for instance for [GHL 87/93], which I have also found useful.

Contents

CHAPTER I

Differential Calculus

We shall recall briefly the notion of derivative and some of its useful properties. As mentioned in the foreword, Chapter VIII of Dieudonné's book or my book on real analysis [La 93] give a self-contained and complete treatment for Banach spaces. We summarize certain facts concerning their properties as topological vector spaces, and then we summarize differential calculus. *The reader can actually skip this chapter* and start immediately with Chapter II if the reader is accustomed to thinking about the derivative of a map as a linear transformation. (In the finite dimensional case, when bases have been selected, the entries in the matrix of this transformation are the partial derivatives of the map.) We have repeated the proofs for the more important theorems, for the ease of the reader.

It is convenient to use throughout the language of categories. The notion of category and morphism (whose definitions we recall in §1) is designed to abstract what is common to certain collections of objects and maps between them. For instance, topological vector spaces and continuous linear maps, open subsets of Banach spaces and differentiable maps, differentiable manifolds and differentiable maps, vector bundles and vector bundle maps, topological spaces and continuous maps, sets and just plain maps. In an arbitrary category, maps are called morphisms, and in fact the category of differentiable manifolds is of such importance in this book that from Chapter II on, we use the word morphism synonymously with differentiable map (or p-times differentiable map, to be precise). All other morphisms in other categories will be qualified by a prefix to indicate the category to which they belong.

I, §1. CATEGORIES

A **category** is a collection of objects $\{X, Y, \dots\}$ such that for two objects X, Y we have a set $\mathrm{Mor}(X, Y)$ and for three objects X, Y, Z a mapping (composition law)

$$\mathrm{Mor}(X, Y) \times \mathrm{Mor}(Y, Z) \to \mathrm{Mor}(X, Z)$$

satisfying the following axioms:

CAT 1. *Two sets* $\mathrm{Mor}(X, Y)$ *and* $\mathrm{Mor}(X', Y')$ *are disjoint unless* $X = X'$ *and* $Y = Y'$, *in which case they are equal.*

CAT 2. *Each* $\mathrm{Mor}(X, X)$ *has an element* id_X *which acts as a left and right identity under the composition law.*

CAT 3. *The composition law is associative.*

The elements of $\mathrm{Mor}(X, Y)$ are called **morphisms**, and we write frequently $f: X \to Y$ for such a morphism. The composition of two morphisms f, g is written fg or $f \circ g$.

A **functor** $\lambda: \mathfrak{A} \to \mathfrak{A}'$ from a category $\mathfrak{A}$ into a category $\mathfrak{A}'$ is a map which associates with each object X in $\mathfrak{A}$ an object $\lambda(X)$ in $\mathfrak{A}'$, and with each morphism $f: X \to Y$ a morphism $\lambda(f): \lambda(X) \to \lambda(Y)$ in $\mathfrak{A}'$ such that, whenever f and g are morphisms in $\mathfrak{A}$ which can be composed, then $\lambda(fg) = \lambda(f)\lambda(g)$ and $\lambda(\mathrm{id}_X) = \mathrm{id}_{\lambda(X)}$ for all X. This is in fact a covariant functor, and a contravariant functor is defined by reversing the arrows (so that we have $\lambda(f): \lambda(Y) \to \lambda(X)$ and $\lambda(fg) = \lambda(g)\lambda(f)$).

In a similar way, one defines functors of many variables, which may be covariant in some variables and contravariant in others. We shall meet such functors when we discuss multilinear maps, differential forms, etc.

The functors of the same variance from one category $\mathfrak{A}$ to another $\mathfrak{A}'$ form themselves the objects of a category $\mathrm{Fun}(\mathfrak{A}, \mathfrak{A}')$. Its morphisms will sometimes be called **natural transformations** instead of functor morphisms. They are defined as follows. If λ, μ are two functors from $\mathfrak{A}$ to $\mathfrak{A}'$ (say covariant), then a natural transformation $t: \lambda \to \mu$ consists of a collection of morphisms

$$t_X: \lambda(X) \to \mu(X)$$

as X ranges over $\mathfrak{A}$, which makes the following diagram commutative for any morphism $f: X \to Y$ in $\mathfrak{A}$:

$$\begin{array}{ccc}
\lambda(X) & \xrightarrow{t_X} & \mu(X) \\
{\scriptstyle \lambda(f)}\downarrow & & \downarrow{\scriptstyle \mu(f)} \\
\lambda(Y) & \xrightarrow[t_Y]{} & \mu(Y)
\end{array}$$

In any category $\mathfrak{A}$, we say that a morphism $f: X \to Y$ is an **isomorphism** if there exists a morphism $g: Y \to X$ such that fg and gf are the identities. For instance, an isomorphism in the category of topological spaces is called a topological isomorphism, or a homeomorphism. In general, we describe the category to which an isomorphism belongs by means of a suitable prefix. In the category of sets, a set-isomorphism is also called a bijection.

If $f: X \to Y$ is a morphism, then a **section** of f is defined to be a morphism $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

I, §2. TOPOLOGICAL VECTOR SPACES

The proofs of all statements in this section, including the Hahn–Banach theorem and the closed graph theorem, can be found in [La 93].

A **topological vector space E** (over the reals **R**) is a vector space with a topology such that the operations of addition and scalar multiplication are continuous. It will be convenient to assume also, as part of the definition, that the space is **Hausdorff**, and **locally convex**. By this we mean that every neighborhood of 0 contains an open neighborhood U of 0 such that, if x, y are in U and $0 \leqq t \leqq 1$, then $tx + (1-t)y$ also lies in U.

The topological vector spaces form a category, denoted by TVS, if we let the morphisms be the continuous linear maps (by linear we mean throughout **R**-linear). The set of continuous linear maps of one topological vector space **E** into **F** is denoted by $L(\mathbf{E}, \mathbf{F})$. The continuous r-multilinear maps

$$\psi: \mathbf{E} \times \cdots \times \mathbf{E} \to \mathbf{F}$$

of **E** into **F** will be denoted by $L^r(\mathbf{E}, \mathbf{F})$. Those which are symmetric (resp. alternating) will be denoted by $L^r_s(\mathbf{E}, \mathbf{F})$ or $L^r_{\mathrm{sym}}(\mathbf{E}, \mathbf{F})$ (resp. $L^r_a(\mathbf{E}, \mathbf{F})$). The isomorphisms in the category TVS are called **toplinear** isomorphisms, and we write $\mathrm{Lis}(\mathbf{E}, \mathbf{F})$ and $\mathrm{Laut}(\mathbf{E})$ for the toplinear isomorphisms of **E** onto **F** and the toplinear automorphisms of **E**.

We find it convenient to denote by $L(\mathbf{E})$, $L^r(\mathbf{E})$, $L^r_s(\mathbf{E})$, and $L^r_a(\mathbf{E})$ the continuous linear maps of **E** into **R** (resp. the continuous, r-multilinear, symmetric, alternating maps of **E** into **R**). Following classical terminology, it is also convenient to call such maps into **R forms** (of the corresponding type). If $\mathbf{E}_1, \ldots, \mathbf{E}_r$ and **F** are topological vector spaces, then we denote by $L(\mathbf{E}_1, \ldots, \mathbf{E}_r; \mathbf{F})$ the continuous multilinear maps of the product $\mathbf{E}_1 \times \cdots \times \mathbf{E}_r$ into **F**. We let:

$$\mathrm{End}(\mathbf{E}) = L(\mathbf{E}, \mathbf{E}),$$

$$\mathrm{Laut}(\mathbf{E}) = \text{elements of } \mathrm{End}(\mathbf{E}) \text{ which are invertible in } \mathrm{End}(\mathbf{E}).$$

The most important type of topological vector space for us is the **Banachable space** (a TVS which is complete, and whose topology can be defined by a norm). We should say **Banach** space when we want to put the norm into the structure. There are of course many norms which can be used to make a Banachable space into a Banach space, but in practice, one allows the abuse of language which consists in saying Banach space for Banachable space (unless it is absolutely necessary to keep the distinction).

For this book, we assume from now on that all our topological vector spaces are Banach spaces. We shall occasionally make some comments to indicate where it might be possible to generalize certain results to more general spaces. We denote our Banach spaces by **E**, **F**, ...

The next two propositions give two aspects of what is known as the **closed graph theorem**..

Proposition 2.1. *Every continuous bijective linear map of* **E** *onto* **F** *is a toplinear isomorphism.*

Proposition 2.2. *If* **E** *is a Banach space, and* $\mathbf{F}_1$, $\mathbf{F}_2$ *are two closed subspaces which are complementary (i.e.* $\mathbf{E} = \mathbf{F}_1 + \mathbf{F}_2$ *and* $\mathbf{F}_1 \cap \mathbf{F}_2 = 0$), *then the map of* $\mathbf{F}_1 \times \mathbf{F}_2$ *onto* **E** *given by the sum is a toplinear isomorphism.*

We shall frequently encounter a situation as in Proposition 2.2, and if **F** is a closed subspace of **E** such that there exists a closed complement $\mathbf{F}_1$ such that **E** is toplinearly isomorphic to the product of **F** and $\mathbf{F}_1$ under the natural mapping, then we shall say that **F** **splits** in **E**.

Next, we state a weak form of the Hahn–Banach theorem.

Proposition 2.3. *Let* **E** *be a Banach space and* $x \neq 0$ *an element of* **E**. *Then there exists a continuous linear map* λ *of* **E** *into* **R** *such that* $\lambda(x) \neq 0$.

One constructs λ by Zorn's lemma, supposing that λ is defined on some subspace, and having a bounded norm. One then extends λ to the subspace generated by one additional element, without increasing the norm.

In particular, every finite dimensional subspace of **E** splits if **E** is complete. More trivially, we observe that a finite codimensional closed subspace also splits.

We now come to the problem of putting a topology on $L(\mathbf{E}, \mathbf{F})$. Let **E**, **F** be Banach spaces, and let

$$A: \mathbf{E} \to \mathbf{F}$$

be a continuous linear map (also called a bounded linear map). We can then define the **norm** of A to be the greatest lower bound of all numbers K such that

$$|Ax| \leqq K|x|$$

for all $x \in \mathbf{E}$. This norm makes $L(\mathbf{E}, \mathbf{F})$ into a Banach space.

In a similar way, we define the topology of $L(\mathbf{E}_1, \ldots, \mathbf{E}_r; \mathbf{F})$, which is a Banach space if we define the norm of a multilinear continuous map

$$A: \mathbf{E}_1 \times \cdots \times \mathbf{E}_r \to \mathbf{F}$$

by the greatest lower bound of all numbers K such that

$$|A(x_1, \ldots, x_r)| \leqq K|x_1| \cdots |x_r|.$$

We have:

Proposition 2.4. *If* $\mathbf{E}_1, \ldots, \mathbf{E}_r$, $\mathbf{F}$ *are Banach spaces, then the canonical map*

$$L(\mathbf{E}_1, L(\mathbf{E}_2, \ldots, L(\mathbf{E}_r, \mathbf{F}), \ldots)) \to L^r(\mathbf{E}_1, \ldots, \mathbf{E}_r; \mathbf{F})$$

from the repeated continuous linear maps to the continuous multilinear maps is a toplinear isomorphism, which is norm-preserving, i.e. a Banach-isomorphism.

The preceding propositions could be generalized to a wider class of topological vector spaces. The following one exhibits a property peculiar to Banach spaces.

Proposition 2.5. *Let* $\mathbf{E}$, $\mathbf{F}$ *be two Banach spaces. Then the set of toplinear isomorphisms* $\mathrm{Lis}(\mathbf{E}, \mathbf{F})$ *is open in* $L(\mathbf{E}, \mathbf{F})$.

The proof is in fact quite simple. If $\mathrm{Lis}(\mathbf{E}, \mathbf{F})$ is not empty, one is immediately reduced to proving that $\mathrm{Laut}(\mathbf{E})$ is open in $L(\mathbf{E}, \mathbf{E})$. We then remark that if $u \in L(\mathbf{E}, \mathbf{E})$, and $|u| < 1$, then the series

$$1 + u + u^2 + \cdots$$

converges. Given any toplinear automorphism w of $\mathbf{E}$, we can find an open neighborhood by translating the open unit ball multiplicatively from 1 to w.

Again in Banach spaces, we have:

Proposition 2.6. *If* $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ *are Banach spaces, then the bilinear maps*

$$L(\mathbf{E}, \mathbf{F}) \times L(\mathbf{F}, \mathbf{G}) \to L(\mathbf{E}, \mathbf{G}),$$

$$L(\mathbf{E}, \mathbf{F}) \times \mathbf{E} \to \mathbf{F},$$

obtained by composition of mappings are continuous, and similarly for multilinear maps.

Remark. The preceding proposition is false for more general spaces than Banach spaces, say Fréchet spaces. In that case, one might hope that the following may be true. Let U be open in a Fréchet space and let

$$f\colon U \to L(\mathbf{E}, \mathbf{F}),$$
$$g\colon U \to L(\mathbf{F}, \mathbf{G}),$$

be continuous. Let γ be the composition of maps. Then $\gamma(f, g)$ is continuous. The same type of question arises later, with differentiable maps instead, and it is of course essential to know the answer to deal with the composition of differentiable maps.

I, §3. DERIVATIVES AND COMPOSITION OF MAPS

A real valued function of a real variable, defined on some neighborhood of 0 is said to be $o(t)$ if

$$\lim_{t \to 0} o(t)/t = 0.$$

Let $\mathbf{E}$, $\mathbf{F}$ be two topological vector spaces, and φ a mapping of a neighborhood of 0 in $\mathbf{E}$ into $\mathbf{F}$. We say that φ is **tangent to** 0 if, given a neighborhood W of 0 in $\mathbf{F}$, there exists a neighborhood V of 0 in $\mathbf{E}$ such that

$$\varphi(tV) \subset o(t)W$$

for some function $o(t)$. If both $\mathbf{E}$, $\mathbf{F}$ are normed, then this amounts to the usual condition

$$|\varphi(x)| \leqq |x|\psi(x)$$

with $\lim \psi(x) = 0$ as $|x| \to 0$.

Let $\mathbf{E}$, $\mathbf{F}$ be two topological vector spaces and U open in $\mathbf{E}$. Let $f\colon U \to \mathbf{F}$ be a continuous map. We shall say that f is **differentiable** at a point $x_0 \in U$ if there exists a continuous linear map λ of $\mathbf{E}$ into $\mathbf{F}$ such that, if we let

$$f(x_0 + y) = f(x_0) + \lambda y + \varphi(y)$$

for small y, then φ is tangent to 0. It then follows trivially that λ is uniquely determined, and we say that it is the **derivative** of f at x_0. We denote the derivative by $Df(x_0)$ or $f'(x_0)$. It is an element of $L(\mathbf{E}, \mathbf{F})$. If