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Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications

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Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications

To my parents Fathi and Mounira
To my wife Faten and our children Hadil, Hiba, and Youssef
and
To my brothers Imed, Aref, and my sister Alyssa

Afif Ben Amar

To my wife Alice and our children Aoife, Lorna, Daniel, and Niamh

Donal O'Regan

Preface

Fixed point theory is a powerful and fruitful tool in modern mathematics and may be considered as a core subject in nonlinear analysis. In the last 50 years, fixed point theory has been a flourishing area of research. In this book, we introduce topological fixed point theory for several classes of single- and multivalued maps. The selected topics reflect our particular interests.

The text is divided into seven chapters. In Chap. 1, we present basic notions in locally convex topological vector spaces. Special attention is devoted to weak compactness, in particular to the theorems of Eberlein–Šmulian, Grothendieck, and Dunford–Pettis. Leray–Schauder alternatives and eigenvalue problems for decomposable single-valued nonlinear weakly compact operators in Dunford–Pettis spaces are considered in Chap. 2. In Chap. 3, we present some variants of Schauder, Krasnoselskii, Sadovskii, and Leray–Schauder-type fixed point theorems for different classes of weakly sequentially continuous (resp. sequentially continuous) operators on general Banach spaces (resp. locally convex spaces). Sadovskii, Furi–Pera, and Krasnoselskii fixed point theorems and nonlinear Leray–Schauder alternatives in the framework of weak topologies and involving multivalued mappings with weakly sequentially closed graph are considered in Chap. 4. The results are formulated in terms of axiomatic measures of weak noncompactness. In Chap. 5, we present some fixed point theorems in a nonempty closed convex of any Banach algebras or Banach algebras satisfying a sequential condition ($\mathcal{P}$) for the sum and the product of nonlinear weakly sequentially continuous operators. We illustrate the theory by considering functional integral and partial differential equations. The existence of fixed points and nonlinear Leray–Schauder alternatives for different classes of nonlinear (ws)-compact operators (weakly condensing, 1-set weakly contractive, strictly quasi-bounded) defined on an unbounded closed convex subset of a Banach space is discussed in Chap. 6. We also discuss the existence of nonlinear eigenvalues and eigenvectors and surjectivity of quasi-bounded operators. In Chap. 7, we present some approximate fixed point theorems for multivalued mappings defined on Banach spaces. Weak and strong topologies play a role here and both bounded and unbounded regions are considered. A method is developed indicating how to use approximate fixed point theorems to prove the existence of approximate Nash equilibria for noncooperative games.

We hope the book will be of use to graduate students and theoretical and applied mathematicians who work in fixed point theory, integral equations, ordinary and partial differential equations, game theory, and other related areas.

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Contents

Chapter 1
Basic Concepts

In this chapter we discuss some concepts needed for the results presented in this book.

1.1 Topological Spaces: Some Fundamental Notions

Let X, Y be arbitrary sets. We use the standard notations $x \in X$ for "x is an element of X," $X \subset Y$ for "X is a subset of Y." The set of all subsets of X is denoted by $P(X)$. Let $\{X_i\}_{i \in I}$ be a family of sets. For the union of this family we use the notation $\bigcup\limits_{i \in I} X_i$ and for intersection the notation $\bigcap\limits_{i \in I} X_i$. If $I = \mathbb{N}$ we have a sequence of sets and we use respectively the notations $\bigcup\limits_{n=1}^{\infty} X_n$ and $\bigcap\limits_{n=1}^{\infty} X_n$. A mapping f of X into Y is denoted by $f : X \longrightarrow Y$. The domain of f is X and the image of X under f is called the range of f. For any $A \subset X$, we write $f(A)$ to denote the set $\{f(x) \ : \ x \in A\} \subset Y$. For any $B \subset Y, f^{-1}(B) = \{x \in X \ : \ f(x) \in B\}$. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are mappings, the composition mapping $x \longmapsto g(f(x))$ is denoted by $g \circ f$. We denote the empty set by $\emptyset$.

Definition 1.1. Let X be any nonempty. A subset τ of $\mathcal{P}(X)$ is said to be a topology on X if the following axioms are satisfied:

1. X and $\emptyset$ are members of τ,
2. the intersection of any two members of τ is a member of τ,
3. the union of any family of members of τ is again in τ

We say that the couple (X, τ) is a topological space. If τ is a topology on X the members of τ are then said to be τ-open subsets of X.

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Definition 1.2. Let (X, τ) be a topological space.

1. The closure of a subset A of X, denoted by $\overline{A}$ is the smallest closed subset containing A.
2. The interior of a subset A of X, denoted by A°, is the largest open subset of A.
3. The boundary of a subset A of X, denoted by ∂A, is the set $\overline{A} \setminus A^\circ$.
4. A subset D is dense in a subset A if $D \subseteq A \subseteq \overline{D}$.
5. A limit point or a cluster pointer or an accumulation point of a subset A is a point $x \in X$ such that each neighborhood of x contains at least one point of A distinct from x.
6. A subset A of X is compact if, for each open covering of A, there exists a finite subcovering. The set A is relatively compact if $\overline{A}$ is compact.
7. The space is locally compact if, for each $x \in X$, there is a neighborhood V_x of x such that $\overline{V_x}$ is compact.
8. A subset A of X is countably compact if, for each countable open covering of A, there is a finite subcovering.

Definition 1.3. A direct set is a nonempty set I with a relation $\leq$ such that

1. $\alpha \leq \alpha$ for all $\alpha \in I$,
2. if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$,
3. for each pair α, β of elements of I, there is $\gamma_{\alpha,\beta}$ such that $\alpha \leq \gamma_{\alpha,\beta}$ and $\beta \leq \gamma_{\alpha,\beta}$.

Definition 1.4. Let (X, τ) be a topological space and I be a directed set. A function x from I into X is said to be a net in X. The expression $x(i)$ is usually denoted by x_i, and the net itself is denoted by $\{x_i\}_{i\in I}$. The set I is the index set for the net.

Definition 1.5. Let (X, τ) be a topological space. A net $\{x_i\}_{i\in I}$ is said to be convergent to a point $x_* \in X$ if for any neighborhood V of x_*, there exists an index $i_V \in I$ such that for any $i \in I$ satisfying $i_V \leq i$, we have that $x_i \in V$. If a net $\{x_i\}_{i\in I}$ is convergent to x_*, we write $\lim_{i\in I} x_i = x_*$

Remark 1.1. It is known that a subset A of X is closed, if and only if for any net $\{x_i\}_{i\in I}$ in A the condition $\lim_{i\in I} x_i = x_0$ implies $x_0 \in A$.

Definition 1.6. Let (X, τ_1), (Y, τ_2) be topological spaces and let $f : X \longrightarrow Y$ be a mapping. We say that f is continuous at a point $x \in X$, if for each τ_2-neighborhood V of $y = f(x)$, $f^{-1}(V)$ is a τ_1-neighborhood of x. If f is continuous at any $x \in X$, then in this case we say that f is continuous on X.

Definition 1.7. Let (X, τ) a topological space:

1. The space X is T_0 if for each pair of distinct points in X, at least one has a neighborhood not containing the other.
2. The space X is T_1 if, for each pair of distinct points in X, each has a neighborhood not containing the other.
3. The space is T_2 or Hausdorff or separated if, for each pair of distinct points x and y, there are disjoint neighborhoods V_x and V_y of x, and y, respectively.

4. The space is T_3 is regular if it is T_1 and, for each x and each closed subset F not containing x, there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.
5. The space X is $T_{3\frac{1}{2}}$ or completely regular or Tychonoff if it is Hausdorff and, for each x and each closed F of X not containing x, there is a continuous function $\alpha : X \longrightarrow [0,1]$ such that $\alpha(x) = 0$ and $\alpha(y) = 1$ for each $y \in F$. In other words, X is completely regular if $C(X,[0,1])$ separates points from closed sets in X. Since singletons are closed in X, we deduce that $C(X,[0,1])$, also separates points in X.
6. The space is T_4 or normal if it is Hausdorff and, for each disjoint closed subsets $F_1, F_2 \subset X$, there are disjoint open subsets V_1 and V_2 such that $F_1 \subset V_1$ and $F_2 \subset V_2$.

Lemma 1.1 (Urysohn). *If F_1 and F_2 are disjoint closed sets in a normal space X, then there is a continuous function $\alpha \in C(X,[0,1])$ such that $\alpha = 0$ on F_1 while $\alpha = 1$ on F_2.*

Theorem 1.1 (Tietze's Extension). *If F is a closed subset of a normal space X, then each continuous function $\alpha \in C(F,[0,1])$ extends to a continuous function $\tilde{\alpha} \in C(X,[0,1])$ on all of X.*

Remark 1.2. From Urysohn's lemma, every normal space is completely regular. Thus, metric spaces and compact Hausdorff are completely regular.

Proposition 1.1. *Let X be a completely regular space. Let F_1, F_2 be disjoint subsets of X, with F_1 closed and F_2 compact. Then there exists a continuous function $\alpha : X \longrightarrow [0,1]$ such that $\alpha \equiv 0$ throughout F_1 and $\alpha \equiv 1$ throughout F_2.*

1.2 Normed Spaces and Banach Spaces

All linear spaces considered in this section are supposed to be over a field $\mathbb{K}$, which can be $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.8. Given a linear space X and a topology τ on X. X is called a topological vector space if the following axioms are satisfied:

(1) $(x,y) \longrightarrow x + y$ is continuous on $X \times X$ into X.
(2) $(\lambda, x) \longrightarrow \lambda x$ is continuous on $\lambda \times X$ into X.

Remark 1.3. Note that we can extend the notion of Cauchy sequence, and therefore of completeness, to a topological vector space : a sequence x_n in a topological vector space is Cauchy if for neighborhood U of θ there exists N such that $x_m - x_n \in U$ for all $m, n \geq N$.

An important class of topological vector spaces is the class of normed vector spaces.

Definition 1.9. Let X be a linear space. A norm on X is a map $\|.\| : X \longrightarrow [0, \infty)$ such that

1. $\|x\| = 0 \Longleftrightarrow x = 0 \qquad (x \in X)$,
2. $\|\lambda x\| = |\lambda| \, \|x\| \qquad (\lambda \in \mathbb{K}, x \in X)$,
3. $\|x + y\| \le \|x\| + \|y\| \qquad (x, y \in X)$.

A linear space equipped with a norm is called a normed space.

Proposition 1.2. *Let $(X, \|.\|)$ be a normed space. Then the mapping*

$$d : X \times X \longrightarrow [0, \infty), (x, y) \longmapsto \|x - y\|$$

is a metric. We may thus speak of convergence, etc., in normed spaces.

Remark 1.4. Let $(X, \|.\|)$ be a normed space. The sets $B(\theta, 1) = \{x \in X : \|x\| < 1\}$ and $\overline{B}_1(\theta) = \{x \in X : \|x\| \le 1\}$ are the open unit ball and the closed unit ball of X, respectively.

Definition 1.10. A normed space X is called a *Banach space* if the corresponding metric space is complete, i.e., every Cauchy sequence in X converges in X.

Now, we discuss some important properties of the first and second duals of a normed space.

Definition 1.11. The topological dual X^* of a normed space $(X, \|.\|)$ is a Banach space. The operator norm on X^* is also called the dual norm, also denoted by $\|.\|$. That is

$$\|\phi\| = \sup_{\|x\| \le 1} |\phi(x)| = \sup_{\|x\| = 1} |\phi(x)| .$$

The topological dual of X' is called the second dual (or the double dual) of X and is denoted by X^{**}. The normed space X can be embedded isometrically in X^{**} in a natural way. Each $x \in X$ gives rise to a norm-continuous linear functional

$$\hat{x}(\phi) = \phi(x) \quad \text{for each } \phi \in X^*.$$

Lemma 1.2. *For each $x \in X$, we have $\|\hat{x}\| = \|x\| = \max_{\|\phi\| \le 1} |\phi(x)|$, where $\|\hat{x}\|$ is the operator norm of $\hat{x}$ as a linear functional on the normed space X^*.*

Corollary 1.1. *The mapping $x \longmapsto \hat{x}$ from X into X^{**} is a linear isometry (a linear operator and an isometry), so X can be identified with a subspace $\hat{X}$ of X^{**}.*

When the linear isometry $x \longmapsto \hat{x}$ from a Banach space X into its double dual X^{**} is surjective, the Banach space is called reflexive. That is, we have the following definition.

Definition 1.12. A space X is called reflexive if $X = \hat{X} = X^{**}$.

1.3 Convex Sets

We start with some basic definitions and a few observations.

Definition 1.13. Let X be a linear space. A subset S of X is said to be **convex** if and only if $\lambda x + (1-\lambda)y \in S$ for every $x, y \in S$ and $\lambda \in [0, 1]$. That is, a convex set is one that contains all points on any "line segment" joining two of its members.

Lemma 1.3. *In any linear space*

1. *The sum of two convex sets is convex*
2. *Scalar multiples of convex sets are convex*
3. *A set S is convex if and only if $\alpha S + \beta S = (\alpha + \beta)S$ for all nonnegative scalars α and β.*
4. *The intersection of an arbitrary family of convex sets is convex.*
5. *In a topological vector space, both the interior and the closure of a convex set are convex.*

Definition 1.14. Let S be any set in a linear space X, and let $\mathcal{S}$ be the class of all convex subsets of X that contains S. We have $\mathcal{S} \neq \emptyset$ since $X \in \mathcal{S}$. Then, $\bigcap \mathcal{S}$ is a convex set in X which, obviously, contains S. Clearly, this set is the smallest (that is, $\supseteq -$ minimum) subset of X that contains S-it is called the **convex hull** of S and denoted by $co(S)$.

Remark 1.5. $S = co(S)$ iff S is convex.

Note

$$co(S) = \{\sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0 \text{ and } x_i \in S \text{ for all } i \leq n \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

1.3.1 Cones

Definition 1.15. A nonempty subset C of linear space X is called a **convex cone** if it satisfies the following properties:

1. C is a convex set.
2. From $x \in C$ and $\lambda \geq 0$, it follows that $\lambda x \in C$.
3. From $x \in C$ and $-x \in C$, it follows that $x = \theta$

A cone can be characterized by 3) together with

$$x, y \in C \quad \text{and} \quad \lambda, \mu \geq 0 \text{ imply } \lambda x + \mu y \in C.$$

Examples 1.1. 1. The set $\mathbb{R}^n_+$ of all vectors $x = (\xi_1, \ldots, \xi_n)$ with nonnegative components is a cone in $\mathbb{R}^n$.

2. The set C_+ of all real continuous functions on $[a, b]$ with only nonnegative values is a cone in the space $C[a, b]$.

Remark 1.6. The set $C \subset l^p (1 \leq p < \infty)$, consisting of all sequences $(\xi_n)_{n \geq 1}$, such that for some $a > 0$

$$\sum_{n=1}^{\infty} |\xi_n|^p \leq a$$

is a convex set in l^p, but obviously, not a cone.

1.3.2 Ordered Vector Spaces

Definition 1.16. If a cone C is fixed in a linear space X, then an **order** can be introduced for certain pairs of vectors in X. Namely, if $x - y \in C$ for some $x, y \in X$ then we write $x \geq y$ or $y \leq x$ and say x is greater than or equal to y or y is smaller than or equal to x. The pair (X, C) is called an **ordered vector space** or a vector space **partially ordered** by the cone C. An element x is called **positive**, if $x \geq 0$ or, which means the same, if $x \in C$ holds. Moreover

$$C = \{x \in X : x \geq 0\}.$$

Remark 1.7. We consider the linear space $\mathbb{R}^2$ ordered by its first quadrant as the cone $C = \mathbb{R}^2_+$. Considering the vectors $x = (1, -1)$ and $y = (0, 2)$, neither the vector $x - y = (1, -3)$ nor $y - x = (-1, 3)$ is in C, so neither $x \geq y$ nor $x \leq y$ holds. An ordering in a linear space, generated by a cone, is always only a partial ordering.

It can be shown that the binary relation $\geq$ has the following properties:

1. $x \geq x \; \forall x \in X$ (reflexivity).
2. $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity)
3. $x \geq y$ and $\alpha \geq 0, \alpha \geq 0, \alpha \in \mathbb{R}$, imply $\alpha x \geq \alpha y$.
4. $x_1 \geq y_1$ and $x_2 \geq y_2$ imply $x_1 + x_2 \geq y_1 + y_2$.

Example 1.1. In the real space $C[a, b]$ we define the natural order $x \geq y$ for two functions x and y by $x(t) \geq y(t), \forall t \in [a, b]$. Then $x \geq 0$ if and only if x is a nonnegative function in $[a, b]$. The corresponding cone is denoted by C_+.

1.3.3 Vector Lattices

Definition 1.17. An ordered vector space X is called a **vector lattice** or **linear lattice** or **Riesz space**, if for two arbitrary elements $x, y \in X$ there exist an element $z \in X$ with the following properties:

1. $x \leq z$ and $y \leq z$,
2. if $t \in X$ with $x \leq t$ and $y \leq t$, then $z \leq t$.

Such an element z is uniquely determined, is denoted by $x \vee y$, and is called the **supremum** of x and y (more precisely: supremum of the set consisting of the elements x and y)

In a vector lattice, there also exists the infimum for any x and y, which is denoted by $x \wedge y$.

Definition 1.18. A vector lattice in which every nonempty subset X that is order bounded from above has a supremum (equivalently, if every nonempty subset that is bounded from below has an infimum) is called a Dedekind or a K-space (Kantorovich space).

Example 1.2. The space $C[a, b]$ is a vector lattice.

Remark 1.8. For an arbitrary element x of a vector lattice X, the elements $x_+ = x \vee \theta$, $x_- = (-x) \vee \theta$ and $|x| = x_+ + x_-$ are called the positive part, negative part, and modulus of the element x, respectively. For every element $x \in X$ the three element $x_+, x_-, |x|$ are positive.

1.3.4 Ordered Normed Spaces

Definition 1.19. Let X be normed space with the norm $\|.\|$. A cone $X_+ \subset X$ is called a **solid**, if X_+ contains a ball (with positive radius), or equivalently, X_+ contains at least one interior point.

A cone X_+ is called **normal** if the norm in X is **semimonotonic**, i.e., there exists a constant $M > 0$ such that

$$0 \leq x \leq y \Longrightarrow \|x\| \leq M\|y.\|$$

A cone is called **regular** if every monotonically increasing sequence which is bounded above

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq z$$

is a Cauchy sequence in X. In a Banach space every closed regular cone is normal.

Examples 1.2. 1. The usual cones are solid in the space $\mathbb{R}$, $C[a,b]$, but in the spaces $L^p([a,b])$ and $l^p(1 \leq p < \infty)$ they are not solid.

2. The cones of the vectors with nonnegative components and the nonnegative functions in the spaces $\mathbb{R}^n$, c_0, l^p and L^p, respectively, are normal.
3. The cones in $\mathbb{R}^n$, l^p and L^p are regular.

1.3.5 Normed Vector Lattices and Banach Lattices

Definition 1.20. Let X be a vector lattice, which is a normed space at the same time. X is called a **normed lattice** or **normed vector lattice**, if the norm satisfies the condition

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\| \quad \forall\, x, y \in X \quad \text{(monotonicity of the norm).}$$

A complete (with respect to the norm) normed lattice is called a **Banach lattice**.

Example 1.3. The spaces $C[a,b]$, L^p and l^p are Banach lattices.

Definition 1.21. Let S be a subset of a normed space X. The **closed convex hull** of S denoted by $\overline{co}(S)$, is defined as the smallest (that is, $\supseteq$ −minimum) closed and convex subset of X that contains S.

Let X be a normed space. Note

$$\overline{co}(S) := \bigcap\{A \in \mathcal{P}(X) : A \text{ is closed in } X, \text{ it is convex, and } S \subseteq A\}.$$

(Note, $\overline{co}(\emptyset) = \emptyset$.)

Clearly, we can view $\overline{co}(.)$ as a self-map on 2^X. Every closed and convex subset of X is a fixed point of this map, and $\overline{co}(S)$ is a closed and convex set for any $S \subseteq X$.

We have this following useful formula

Proposition 1.3. *Let X be a normed space. Then*

$$\overline{co}(S) = \overline{co(S)} \qquad \textit{for any } S \subseteq X.$$

Proof. Since $\overline{co(S)}$ is convex, it is a closed and convex subset of X that contains S, so $\overline{co}(S) \subseteq \overline{co(S)}$. The $\supseteq$ part follows from the fact that $\overline{co}(S)$ is a closed set in X that includes $co(S)$. ■