COMPUTER ANALYSIS OF STRUCTURES

MATRIX STRUCTURAL ANALYSIS STRUCTURED PROGRAMMING

Siegfried M. Holzer

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PREFACE

This book was written with two principal *objectives for the students* in mind: (1) to acquire a precise understanding of the *matrix displacement method* and its underlying concepts and principles; (2) to develop *well-structured programs* for the analysis of skeletal structures by the matrix displacement method. The displacement method was selected as the method of analysis because of its intrinsic modularity, good numerical properties, and popularity—matrix and finite element analysis programs are generally based on the displacement method. Structured programming is emphasized because it provides a systematic process for creating correct programs.

Students have demonstrated that they can use this programming knowledge to write special-purpose programs, such as computer-aided design programs and finite element programs.

The history of the development of matrix and finite element methods of analysis, motivated by the computer, is traced, for example, by Martin and Carey (1973). An overview of structured programming is provided in Bates (1976).

In the first three chapters, the matrix displacement method is presented in a form suitable for programming. The matrix displacement method is extended to special topics in Chapter 4 and to space structures in Chapter 5. Chapter 6 deals with the numerical solution of the system equations. Chapter 7 is concerned with structured programming. Five appendices include elementary methods of analysis, principles of analytical mechanics, and mathematical tools.

Discrete element models are formulated in Chapter 1 by three approaches: (1) by the solution of differential equations; (2) by force–deformation formulas,[1] and (3) by the finite element method. The students should adopt the approach that provides a natural link to their background. For example, students who have had a course in the mechanics of deformable bodies but who have not had a basic course in structural analysis should take the first approach. Students who have completed a traditional junior-level course in structural analysis might prefer the second approach, in which the extensional force–deformation relation and the slope–deflection equations are combined with conditions of equilibrium to construct the models for truss and beam elements. The finite element approach is recommended for students with a basic understanding of the matrix displacement method and the principle of virtual work.

Chapter 2 paves the way for the matrix displacement method appro priate for program development. The central task is the formulation o conditions of compatibility and equilibrium without a visual reference tc the structure. For this purpose, the member code matrix is introducec (Section 2.4).

In Chapter 3 the matrix displacement method is formulated on the basis of the member code matrix, and it is illustrated for continuous beams, frames, and trusses. Joint loads and member loads are considered.

Chapter 4 illustrates how special features can be incorporated in the matrix displacement analysis. The topics covered can be divided into three groups: (1) the reduction of the degrees of freedom of an assembly of elements by utilizing symmetry, by introducing internal constraints, and by condensation; (2) the formulation of various element actions, such as geometric imperfections, temperature changes, and unit displacements imposed in the construction of influence lines by the Müller–Breslau principle; and (3) the formulation of assemblies with distinct elements, internal releases, and distinct joint reference frames.

In Chapter 5, the matrix displacement method is extended to space structures: space trusses, space frames, and grids.

Chapter 6 is concerned with the numerical solution of the system equations. It includes a literature review of solution techniques, a discussion of storage schemes for fixed and variable band matrices, the formulation of direct solution methods with reference to special structural analysis techniques for symmetric, positive definite band matrices, algorithms for fixed and variable band solvers and references to computer programs, the frontal solution technique and references to computer programs, and a study of solution errors and methods of error detection and error control.

Chapter 7 is concerned with structured programming. It includes discussions of the aims of structured programming, control structures, methods

[1] In the presentation, the first two approaches are not separated. The beginning of the second approach, which is contained in the first approach, is stated in Section 1.1.

of modularization, programming and coding languages, program correctness, and program efficiency. The principles of structured programming are applied in the design of a program for the matrix displacement analysis of plane frames. The program structure is represented by a tree chart, and the subprograms are described by structured flow charts (Nassi–Schneiderman diagrams) with lists of input and output arguments. FORTRAN 77 and the FORTRAN version of the WATFIV compiler are used to illustrate the coding of several subprograms.[2] A variety of programming problems is presented that includes the completion of the frame program (coding and testing) and extensions based on Chapters 4–6.

In Appendix A, fixed-end force formulas are presented for various element actions. In addition, displacement–deformation relations are derived that permit us to apply the moment–area method without a sketch of the deformed configuration. In Appendix B, the slope–deflection method is presented in a form that facilitates the transition to the matrix displacement method. Appendix C provides a comprehensive treatment of coordinate transformations. Appendix D is concerned with the principle of virtual work. In Appendix E, the imposition of joint constraints at the system level is addressed.

COURSE DESIGN. After the matrix displacement method in Chapters 1–3 has been studied, the remaining chapters can be studied in any order. It is recommended, however, to follow the formulation of the matrix displacement method in Chapter 3 with the program development as indicated in the course structure.

Upon the completion of the frame program in Chapter 7, the study of the material in Chapters 4–6 can be integrated with desirable program extensions.

The matrix displacement method and program development, with the completion of the frame program as a class project, are appropriate for a single semester of a senior-level course. In our quarter system, the textbook forms the basis for two senior-level courses. In the first course, we teach the matrix displacement method and, depending on the students' progress, one or two topics of Chapter 4. The second course is primarily concerned with program development, the frame program class project, and the matrix displacement analysis of space structures.

In designing and teaching a course, I find the two educational commandments of A. N. Whitehead (1967) reassuring:

> Do not teach too many subjects.
> What you teach, teach thoroughly.

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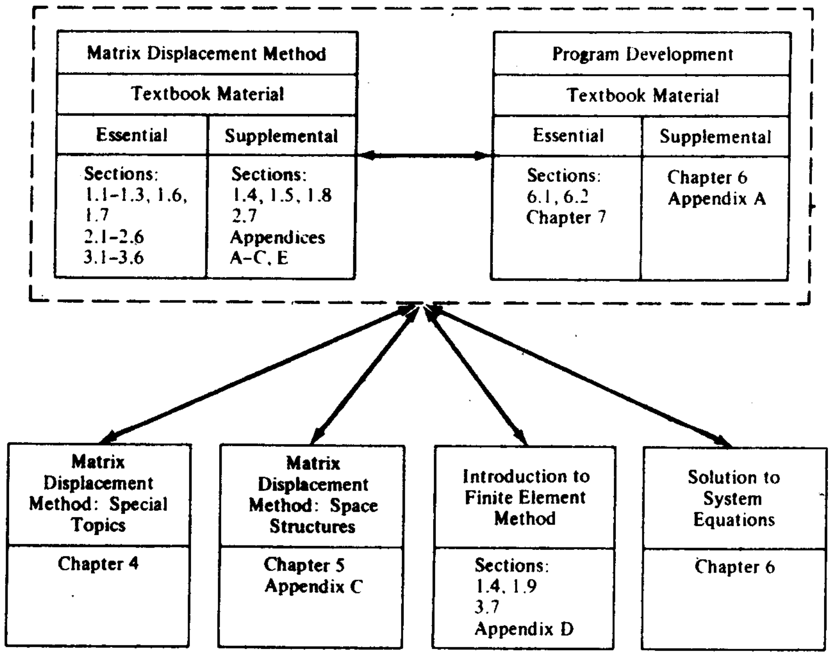
[2] The completed frame program is available to the teacher.



Course Structure

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S.M. Holzer

CONTENTS

CHAPTER

ONE

MATHEMATICAL MODELS OF ELEMENTS

1.1 INTRODUCTION

The prediction of the performance of a structure, which is the role of structural analysis, is generally based on the analysis of mathematical models. The accuracy of this prediction depends on how well the models approximate the behavior of the structure. Accordingly, it is important to know the limitations of the mathematical models used to represent the structure.

Insight into the limitations of a model can be gained from its construction. The first step in the construction of a model is to select generalized displacements, which define the configuration and determine the degrees of freedom of the model. If a model has infinitely many degrees of freedom, it is called a continuum model; otherwise it is called a discrete model. Next, the three basic components of a model, the conditions of compatibility, the conditions of equilibrium, and the constitutive law, are formulated. The synthesis of these components yields the mathematical model.

The primary purpose of this chapter is to formulate discrete element models for the analysis of skeletal structures by the matrix displacement method. The chapter is organized as follows: After some brief discussions of structural analysis and mathematical models (Sections 1.2 and 1.3), the concepts of generalized displacements and forces are defined and illustrated (Section 1.4). A modeling process is introduced and applied to obtain one-dimensional continuum models representing axial, flexural, and torsional deformations (Section 1.5). Discrete element models, which relate element-end displacements to element-end forces through stiffness matrices, are formulated by three approaches: (1) by the solution of continuum models (Sections 1.6 and 1.7); (2) by force–deformation formulas (Sections 1.6 and 1.7)—specifically, the axial deformation formula, Eq. (1.72), the slope–deflection equations, Eqs. (1.86), and the torsional deformation formula, Eq. (1.91); (3) by the finite element method (Section 1.9). It is recommended to select the approach that suits the students' background (see Preface). Special topics of discrete elements are presented in Section 1.8.

1.2 STRUCTURAL ANALYSIS

Structural engineering is concerned with the planning, designing, and building of structures. Structural analysis forms an integral part of the design process. Its function is to predict the behavior of a structure in its environment. This prediction is usually based on mathematical models. Physical models may be used if the reliability of a mathematical model is in doubt.

A model of a structure is defined as a mathematical representation of the behavior of the structure in its environment. It is expressed as an action-response relation. Actions are mathematical models of such environmental factors as loads, prescribed displacements, and temperature changes. The response is a measure of the change in state of the structure. It may be expressed, for instance, by displacements, strains, stresses, and forces.

In essence, structural analysis is concerned with the specification of actions, the construction of models of the structure, and the determination of the response to the imposed actions (Figure 1.1).

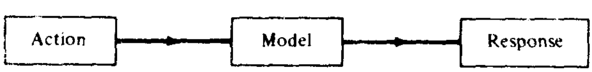
1.3 MATHEMATICAL MODELS

A mathematical model of a structure has three distinct components that represent signficant features of the structure: conditions of compatibility, conditions of equilibrium, and constitutive laws.

The conditions of compatibility reflect geometric properties of a structure, such as continuity of deformations of elements and assemblages of elements, and boundary constraints. In addition, restrictions on deformations are frequently imposed to reduce the three-dimensional structure to a two- or one-dimensional model. For example, the assumption concerning plane sections in the elementary beam theory (Crandall, Dahl, and Lardner, 1978; Freudenthal, 1966; Popov, 1968; Stippes, Wempner, Stern, and Beckett, 1961) transforms the beam into a one-dimensional element.

The conditions of equilibrium express the state of balance of a structure at rest. Newton's law or the principle of virtual work can be applied to

FIGURE 1.1 Analysis process



formulate the conditions of equilibrium. If the structure is in motion, the conditions of equilibrium are replaced by the laws of motion.

The constitutive laws model the behavior of materials. For example, Hooke's law is based on the idealization of a linearly elastic material.

It is important to keep in mind that mathematical models of structures represent idealizations. The results obtained from the analysis of the model can be valid only to the extent that the model approximates the behavior of the stucture.

1.4 GENERALIZED DISPLACEMENTS AND FORCES

Generalized Displacements

The formulation of a mathematical model of a structure centers on the selection of parameters that define the configuration of the model. The *configuration* is characterized by the simultaneous locations of all material points. The number of independent parameters required to define the configuration represents the *degrees of freedom* of the model. These parameters are called the *generalized displacements* (or generalized coordinates;[1] Langhaar, 1962) of the model.

In engineering analysis, the configuration of a model is generally described relative to its *initial state*, a reference configuration in which the model is not subjected to actions. Specifically, the configuration is defined by the displacements of each point from its initial position. This is illustrated in the following examples for one-dimensional elements without reference to actions that may correspond to these configurations.

Example 1. Consider the rigid bar in Figure 1.2, whose initial configuration coincides with the x axis of the rectangular frame of reference. Thus, the initial configuration is defined by the set of points $0 \leq x \leq L$. If the bar is confined to the x–y plane, it has three degrees of freedom. There is considerable freedom in the selection of generalized displacements. For example, the position of the bar in the x–y plane can be specified by the displacements—the deflections and rotation—of any initial point of the bar. Let us select the displacements at the a end of the bar, u_a, v_a, and θ_a, as generalized displacements and formulate the configuration in terms of them.

The deflections of the initial point P, located a distance x from the a end of the bar, in the directions of the x and y axes are denoted by $u(x)$ and $v(x)$, respectively. Since the bar is rigid, the distance from the a end of the bar to

[1] This definition is restricted to holonomic models in which the generalized displacements can be varied arbitrarily without violating kinematical constraints. Nonholonomic models are discussed by Lanczos (1970), Langhaar (1962), and Synge and Griffith (1959).

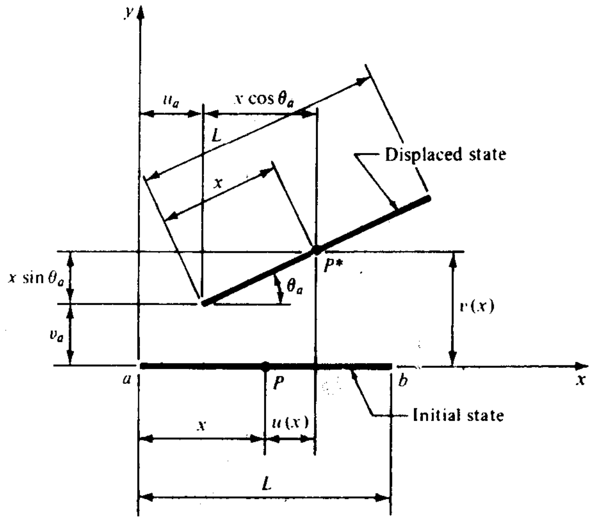


FIGURE 1.2 Rigid bar

the point P^*, the displaced position of point P, remains x. It follows from Figure 1.2 that

$$\left.\begin{aligned} u(x) &= u_a + x(\cos \theta_a - 1) \\ v(x) &= v_a + x \sin \theta_a \end{aligned}\right\} \quad 0 \leq x \leq L \tag{1.1}$$

Equations (1.1) define the displaced position of every initial point of the bar in terms of the generalized displacements.

If the rotation of the bar is infinitesimal, that is, if

$$\theta_a^2 \cong 0 \tag{1.2}$$

relative to unity, we obtain (Thomas, 1956)

$$\begin{aligned} \cos \theta_a &= 1 - \frac{\theta_a^2}{2!} + \frac{\theta_a^2}{4!} - \cdots \cong 1 \\ \sin \theta_a &= \theta_a - \frac{\theta_a^3}{3!} + \frac{\theta_a^5}{5!} - \cdots \cong \theta_a \end{aligned} \tag{1.3}$$

and Eqs. (1.1) become

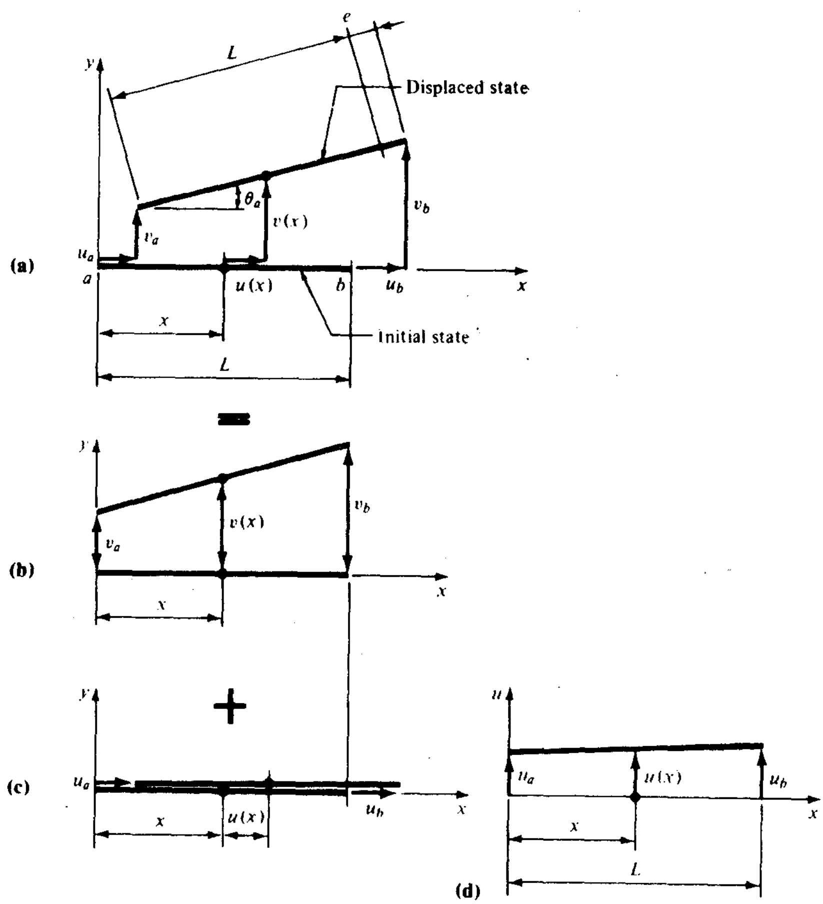
$$\begin{aligned} u(x) &= u_a \\ v(x) &= v_a + x\theta_a \end{aligned} \tag{1.4}$$

Observe that for infinitesimal rotations, the configuration of the bar is a linear function of the generalized displacements. This is characteristic of *linear models* of structures.

Example 2. If the element in Figure 1.2 is free to experience a uniform axial deformation, it becomes the four-degree-of-freedom element shown in Figure 1.3a. Accordingly, the three rigid-body displacements, u_a, v_a, θ_a, and the deformation e represent a set of generalized displacements. However, if the element is part of an assemblage of elements, such as a truss, it is preferable to select the element-end deflections, u_a, v_a, u_b, v_b, as generalized displacements. The formulation of the element configuration in terms of the end deflections is illustrated.

It is convenient to resolve the configuration in Figure 1.3a into component

FIGURE 1.3 Axial deformation element



configurations, representing transverse and axial deflections separately (Figures 1.3b and c). Thus, the transverse deflections in Figure 1.3b are defined by the equation of the line

$$v(x) = v_a + \frac{x}{L}(v_b - v_a) \tag{1.5}$$

which can be expressed as

$$v(x) = \phi_a(x)\, v_a + \phi_b(x)\, v_b \tag{1.6}$$

where

$$\phi_a = 1 - \frac{x}{L}, \quad \phi_b = \frac{x}{L}, \qquad 0 \le x \le L \tag{1.7}$$

Since the axial deformation is uniform, the axial deflection varies linearly as shown in Figure 1.3d (see Problem 1.3). Hence, $u(x)$ can be expressed in the form of Eq. (1.6) as

$$u(x) = \phi_a(x)\, u_a + \phi_b(x)\, u_b \tag{1.8}$$

An alternative formulation of Eqs. (1.6) and (1.8) is based on function interpolation (Section 1.9). Specifically, since $u(x)$ and $v(x)$ must satisfy the boundary conditions

$$u(0) = u_a, \quad u(L) = u_b; \quad v(0) = v_a, \quad v(L) = v_b \tag{1.9}$$

we can express them as first-order polynomials,

$$u = b_0 + b_1 x, \quad v = c_0 + c_1 x \tag{1.10}$$

and determine the coefficients by imposing Eqs. (1.9). This approach is illustrated in the next example.

Example 3. Consider the element in Figure 1.4, whose configuration is defined by the polynomial function

$$v(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \tag{1.11}$$

By definition, the coefficients of Eq. (1.11) represent generalized displacements and the element has four degrees of freedom. The reasons for introducing Eq. (1.11) are that (1) it satisfies the homogeneous differential equation of a beam (Section 1.6), hence, Eq. (1.11) characterizes the configurations of beams subjected to boundary actions; and (2) polynomials form basic building blocks of many interpolation functions.

Analogous to Example 2, let us select the element-end displacements, $v_a, \theta_a, v_b, \theta_b$, as generalized displacements and express the configuration in terms of them. This can be accomplished by imposing the boundary conditions

$$\begin{aligned} v(0) &= v_a, \quad \frac{dv(0)}{dx} = \theta_a \\ v(L) &= v_b, \quad \frac{dv(L)}{dx} = \theta_b \end{aligned} \tag{1.12}$$