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PLANE AND SOLID GEOMETRY

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PREFACE TO THE FIRST EDITION

It is generally conceded that the final aim of mathematical teaching should be not only the acquisition of practical knowledge but that training of the student's mind which gives a distinct gain in mental power. In recognition of this principle nearly all college entrance examinations in geometry require some original work, and most textbooks devote considerable space to exercises. Comparatively little, however, has been done to introduce the student *systematically* to original geometrical work. No teacher of physics or chemistry would ask a student to discover a law without so guiding his work as to enable him to reach the desired result; many textbooks and teachers expect the pupil to invent geometrical proofs and to solve problems, entirely new to him, without offering any assistance further than a knowledge of the well-established theorems of all textbooks. Some writers give a description of the analysis of propositions, which is entirely logical and of great advantage to a person of some mathematical knowledge, but which is usually too abstract to be of any practical value to the beginner. In this book the attempt is made to introduce the student systematically to the solution of geometrical exercises. In the beginning the exercises given in a certain group are of similar kind and related to the preceding proposition; later some general principles are developed which are of fundamental importance for original work, as, for example, the method of proving the equality of lines by means of equal triangles; the method of proving the proportionality of lines by means of similar triangles, etc.; and finally the analyses of theorems and problems are introduced, but in a more concrete form than usual.

The propositions are arranged with the view to obtaining a perfect logical and pedagogical order. An unusually large number of exercises is given, selected with care for the purpose of securing increased mental power.

The general plan and the preparation of the greater part of the book are the work of Dr. Schultze, while that of Dr. Sevenoak has been chiefly editorial.

PREFACE TO THE FIRST REVISED EDITION

The main purpose of the revision of this book has been to emphasize still further and to elaborate in greater detail the principal aim of the original edition, viz., *to introduce the student systematically to original geometric work.* To make the teaching of geometry both disciplinary and informational; to give to the student mental training instead of teaching him mere facts; to develop his power instead of making him memorize, — these are the fundamental aims of this book.

The means employed for this purpose are similar to those used in the first edition. Still greater emphasis, however, has been placed upon the general methods which may be used for the solution of original exercises. The grading and the selection of exercises have been carefully revised. All originals that appeared unfit or too difficult have been eliminated or replaced by simpler and better ones. Topics of fundamental importance, *e.g.* the methods of demonstrating the equality of lines, are represented in greater detail and illustrated by a greater number of exercises than in the first edition.

In addition to these fundamental tendencies, a number of minor improvements have been introduced, among which may be mentioned:

Improved presentation of the regular propositions. Many proofs have been simplified, a more pedagogic sequence of the propositions of Book I has been adopted, Books VI and VII have been considerably simplified, and a number of difficult theorems of minor importance have been omitted or placed in the appendix.

Simplification of the so-called "incommensurable case." As this is a claim that is made by most textbooks, it may be received with some degree of skepticism, but a repeated trial of this new method will reveal its simplicity. For the more conservative teacher, however, who dislikes fundamental changes, the time-honored method is given in the appendix.

The introduction of many applied problems. These problems have been selected and arranged so as to increase the interest of the student, without sacrificing in the least the disciplinary value of the subject. Many such problems are given in the appendix.

The arrangement of the propositions and the terminology are in accord with the best modern usage. Thus, statements and reasons have been separated and placed in parallel vertical columns; the term "congruent" and the corresponding symbol are introduced and applied consistently, etc.

Many of the diagrams have been improved. The construction lines are drawn completely for most problems, graphical modes are employed for pointing out important facts, and many diagrams have been otherwise improved.

Thanks are due to Dr. J. Kahn and Mr. W. S. Schlauch for assistance in reading the proof and for helpful suggestions.

A. S.

August 1, 1913.

PREFACE TO THE PRESENT EDITION

In the twelve years since the first revision of Schultze and Sevenoak's Geometry was made, the progress of teaching and the codifying of mathematical requirements have necessitated some changes in form and content. The present revision follows closely the first revision in organization, but has been completely rewritten with additional material. The noteworthy points are:

1. The text has been made to conform to the "Report of the National Committee on Mathematical Requirements" and to the requirements of the New York State Regents and of the College Entrance Examination Board.

2. Most of the excellent exercises of the previous edition have been kept intact and other exercises, from recent examination questions, have been added.

3. The propositions, especially in Books I and II, have been rather fully developed.

4. Additions have been made in (1) propositions, (2) simple trigonometric functions, (3) tables, (4) short review section on arithmetic and algebra, (5) a sketch of the history of geometry.

5. At the end of the various books and in the appendix, the more difficult propositions have been retained so as to meet the most exacting requirements of any college or technical school.

The authors and publishers are conscious of a great debt of gratitude to many teachers who have aided in this revision by their criticisms and suggestions. While a full list of such

collaborators is impossible, special thanks are due the following: Mr. Stephen Emery, Erasmus Hall High School, Dr. Loring B. Mullen, Girls High School, Miss Josephine D. Wilkin, Jamaica High School, Mr. Philip R. Dean, Evander Childs High School, and Mrs. Jean F. Brown, Hunter College High School, of New York City; Mr. Edmund D. Searls, High School, of New Bedford, Massachusetts; Miss A. Laura Batt, High School, of Somerville, Massachusetts; Mr. Allen H. Knapp, Central High School, and Mr. Harry B. Marsh, Technical High School, of Springfield, Massachusetts; and Miss Anna H. Andrews, Public High School, of Hartford, Connecticut.

E. S.

SUGGESTIONS TO STUDENTS

The student should set to work guided by the following suggestions:

1. Have materials — pencil, paper, ruler, compasses — ready for use.

2. Have a definite time and place to do your work.

3. Do your work *yourself*. Do it *regularly* and *conscientiously*.

4. Frequently review important matters already gone over to fix them firmly in your mind.

5. In a notebook classify your results, stating conditions that make angles equal, lines parallel, triangles equal, etc. Classify formulas.

6. Do not merely skim through the text. Give it thought. Read carefully, and thoroughly digest what you do.

SYMBOLS AND ABBREVIATIONS

$+$ . .	plus, or added to.
$-$ . .	minus, or diminished by.
$=$ . .	equals, or is equivalent to.
$\cong$ . .	congruent.
$\neq$ . .	is not equal to.
$>$ . .	is greater than.
$<$ . .	is less than.
$\therefore$ . .	therefore, or hence.
$\perp$ . .	perpendicular, or is perpendicular to.
$\perp$s . .	perpendiculars.
$\parallel$ . .	parallel, or is parallel to.
$\parallel$s . .	parallels.
$\sim$ . .	is similar to, or similar.
$\angle$ . .	angle.
$\angle$s . .	angles.
$\triangle$ . .	triangle.
$\triangle$s . .	triangles.
▱ . .	parallelogram.
▱s . .	parallelograms.
$\odot$ . .	circle.
ⓢ . .	circles.
⌒ . .	arc; as $\overset{\frown}{AB}$, arc AB.
adj. .	adjacent.

alt.	alternate.
ax.	axiom.
circum. . .	circumference.
comp. . .	complement.
con. . . .	construction.
cor. . . .	corollary.
corr. . . .	corresponding.
def. . . .	definition.
ex.	exercise.
ext. . . .	exterior.
hom. . . .	homologous.
hy.	hypotenuse.
hyp. . . .	hypothesis.
iden. . . .	identity.
int. . . .	interior.
isos. . . .	isosceles.
rt.	right.
st.	straight.
sub. . . .	substitution.
sup. . . .	supplementary, or supplement.

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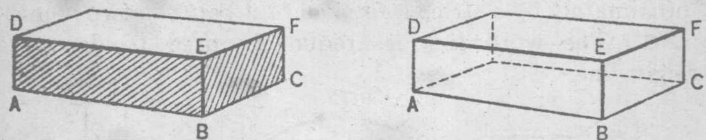
SOLID GEOMETRY

PLANE GEOMETRY

INTRODUCTION

DEFINITIONS

1. A physical body, such as a block of wood or iron, occupies a definite portion of space. The portion of space occupied by a physical body is called a *geometric solid* or a *solid*.

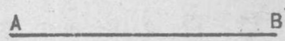


2. DEF. **Solids** are limited portions of space. They have three dimensions: *length*, *breadth*, and *thickness*.

3. DEF. **Surfaces** are the boundaries of solids; as *ABED* or *BEFC*. They have two dimensions: *length* and *breadth*.

The boundary between a window pane and the air is a surface. Obviously such a boundary has no thickness.

4. DEF. **Lines** are the boundaries of surfaces; as *AB*, *AD*. (Figure of § 1.) Lines have but one dimension: *length*.

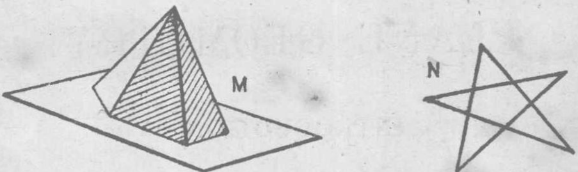


Thus, the annexed black line *AB* is not a geometric line, for it has breadth. A true geometric line, however, is represented by the boundary between the black and the white.

5. DEF. **Points** are the boundaries, or the extremities, of lines. They are without dimensions, having position only.

Surfaces may be conceived as existing independently of the solids whose boundaries they form. In like manner, lines and points may exist independently in space.

6. Def. A **geometric figure** is a point, line, surface, or solid, or any combination of any or all of these; as *M* or *N*.



A *rectilinear figure* is a figure composed of straight lines only.

7. Def. **Geometry** is the science that treats of the properties of geometric figures.

8. The simplest line is a **straight line.** It is *represented* approximately by a string stretched taut between two points; as *AB*. The word *line* is frequently used to denote a straight line.

The notion of a straight line is such a simple and fundamental one that it is practically impossible to give a good definition of it.

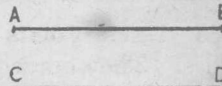
9. Def. A **curved line** is a line no portion of which is straight; as *CD* in § 8.

10. Def. A **broken line** is a line composed of different successive straight lines; as *EF* in § 8.

No *two successive* parts of a broken line lie in the same straight line.

11. The expression, straight line, is used to denote both an unlimited straight line and a part of such line.

A *line of definite length,* also called a segment, or line-segment, is represented by a line whose ends are marked; as *AB*. The length of this line is also called the **distance** between *A* and *B*.



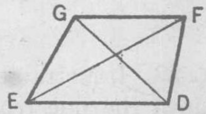
A line whose ends are not marked represents a line of indefinite length; as *CD*.

12. The **direction** of the line *AB* means the direction from *A* toward *B*; of *BA*, the direction from *B* toward *A*.

13. To **produce** the line *AB* means to prolong it through *B*; to **produce** *BA* means to prolong it through *A*.

14. DEF. A **plane surface**, or a **plane**, is a surface such that a straight line joining any two of its points lies entirely in the surface.

15. DEF. A **plane figure** is a geometric figure, all of whose points lie in the same plane; as *EDFG*.



16. DEF. **Plane Geometry** treats of plane figures only.

17. DEF. **Solid Geometry** treats of figures which are not plane. **Spherical Geometry** is a surface geometry.

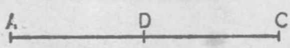
18. When one figure can be placed upon another so that every point of each figure lies upon some point of the other, the figures are said to **coincide**.

19. DEF. **Congruent figures** are those that can be made to coincide.

For reasons that will appear later congruent lines are frequently called *equal lines*. Similarly, congruent angles are usually called ***equal angles***. (See note, p. 205.)

20. **Proof by superposition** is the method of proving the congruence of two figures by making them coincide.

21. To **bisect** a line means to divide it into two equal parts.



Thus, AC is bisected if $AD = DC$. We assume that every line-segment (AC) has one and only one bisector. This bisector is a point; as D.

Ex. 1. What is the path of a moving point?

Ex. 2. What geometric figure is, in general, generated by a moving line? By a moving surface?

Ex. 3. Can a straight line move so that its path is not a surface?

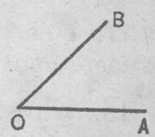
Ex. 4. How does a stone cutter use the straight edge to determine whether a surface is plane?

Ex. 5. What kind of surface is represented by each wall of a room?

Ex. 6. What kind of surface is represented by the outer surface of a gas pipe?

ANGLES

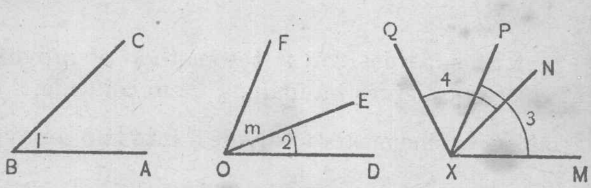
22. If a straight line *OA* revolves about one of its points *O* until the line reaches the position *OB*, then the amount of this rotation is called *the angle AOB*. Obviously the amount of rotation, and hence the angle, does not depend upon the length of the line which rotates.



The lines *OA* and *OB* are called the **sides** and the point *O* the **vertex** of the angle *AOB*.

We may define an *angle* to be the figure formed by two rays, or half lines, proceeding from a common point.

23. Notation. If three letters are used to denote an angle, the vertex letter should be read between the others; as angle *ABC*, angle *EOF*. A single letter at the vertex denotes the



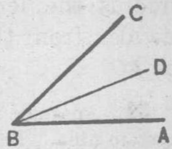
largest angle at this vertex (if there be several at this point). Thus, angle *DOF* may be read "angle *O*," angle *ABC* may be read "angle *B*."

Frequently an angle is also designated by a number, or italic letter, placed within it; as angle 1, **angle 2**, **angle *m*.**

Angle *FOD* is the **sum** of angles 2 and *m*. Angle 2 is the **difference** between angles *FOD* and *m*.

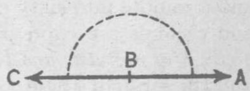
Often a curve is drawn to point out more clearly which angle is meant; as angle 2, and angle 3. An arc placed close to a number shows which angle is designated. Thus, angle *MXP* may be read "angle 3," and angle *NXQ* may be read "angle 4."

24. To **bisect** an angle means to divide it into two equal parts. We assume every angle has one and only one bisector.



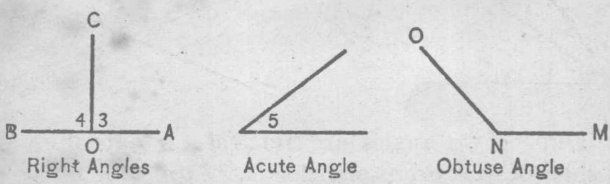
Thus, *BD* bisects angle *ABC*, if angle *ABD* = angle *DBC*. *BD* is called the **bisector** of angle *B*.

25. Def. A **straight angle** is an angle whose sides lie in the same straight line but extend in opposite directions; as *ABC*.



26. Def. A **right angle** is an angle equal to one half a straight angle.

Thus, if *OC* bisects the straight angle *AOB*, angle 3 and angle 4 are right angles.

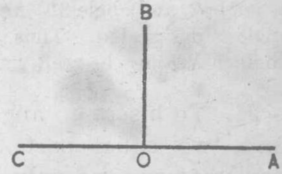


27. Def. An **acute angle** is an angle less than a right angle; as angle 5.

28. Def. An **obtuse angle** is an angle greater than a right angle, but less than a straight angle; as angle *MNO*.

29. Def. Acute and obtuse angles are called **oblique angles**.

30. Def. Two lines are **perpendicular** to each other if they meet at right angles; as *AC* and *BO*.

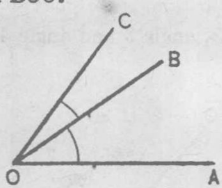


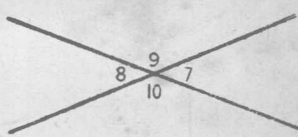
The point of meeting (*O*) is the **foot** of the perpendicular.

The **distance** of a point from a line is the length of the perpendicular from the point to this line; as *BO*.

31. An angle is **measured** by finding how many times it contains a certain unit. The usual unit is the **degree**, or one ninetieth ($\frac{1}{90}$) of a right angle. A degree is divided into sixty equal parts called **minutes**, and a minute into sixty equal parts called **seconds**. Degrees, minutes, and seconds are expressed by symbols, as $6° \; 50' \; 12''$. Read *six degrees, fifty minutes, and twelve seconds.* Other units are the right angle and the straight angle.

32. Def. **Adjacent angles** are two angles that have a *common vertex*, and a *common side between them;* as angles *AOB* and *BOC*.



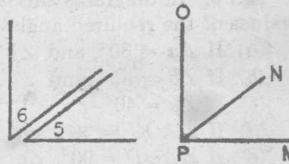


33. Def. Two angles are **vertical** (or **opposite**) **angles** if the sides of each are prolongations of the sides of the other back through their common vertex; as angles 7 and 8, or angles 9 and 10.

34. Def. We can **add** two angles by placing them so that they are adjacent. The angle formed by the two sides not common is called their **sum**.

35. Def. Two angles are **complementary** if their sum equals a right angle.

Each is then called the *complement* of the other. Angles 5 and 6, or *MPN* and *NPO*, are complementary.



36. Two angles are **supplementary** if their sum equals a straight angle (or two right angles).

Each angle is then called the supplement of the other. Angles 1 and 2, or angles 3 and 4, are supplementary.

Ex. 1. How many degrees are in a right angle? In a straight angle? In one half a right angle?

Ex. 2. What is the angle made by the two hands of a clock at three o'clock? At six o'clock? At two o'clock? At five o'clock?

Ex. 3. What is the angle made by the hands of a clock at 1 P.M.? At 2:30 P.M.? At 5:30 P.M.?

Ex. 4. Over an angle of how many degrees does a spoke of a wheel sweep when the wheel makes $\frac{1}{4}$ of a revolution? $\frac{1}{8}$ of a revolution? 2 revolutions?

Ex. 5. How large is each angle at the center if a pie is divided into 5 equal parts? 6 equal parts?

Ex. 6. What angle is formed by lines drawn north and northeast? Towards S. and S.E.? Towards N.W. and S.W.?

Ex. 7. Over what angle does the large hand of a watch sweep in 10 min.? 15 min.? 30 min.? 45 min.? 1 hr.?

Ex. 8. In diagram for Ex. 9 read by three letters: $\angle a$, $\angle b$, $\angle c$, $\angle d$, $\angle(a+b)$, $\angle(b+c+d)$.